Decay probabilities in the multichannel case

Francesco Giacosa
UJK Kielce – Goethe U Frankfurt

4th Jagiellonian Symposium on
Advances in Particle Physics and Medicine
Krakow – 10-15/7/2022

Outline

1. General discussion of the decay law
2. Non-exponential decay: experiments
3. Multichannel case in QM
4. Short digression: double-delta potential
5. Multichannel case in QFT
6. Related topics and conclusions

Physics Letters B 831 (2022) 137200

ELSEVIER

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Uniwersytet Jana Kochanowskiego w Kielcach

Multichannel decay law

Francesco Giacosa [a,b,*]

[a] Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406, Kielce, Poland
[b] Institute for Theoretical Physics, J. W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt, Germany

Check for updates

A R T I C L E   I N F O

Article history:
Received 4 November 2021
Received in revised form 24 April 2022
Accepted 23 May 2022
Available online 26 May 2022
Editor: A. Ringwald

A B S T R A C T

It is well known, both theoretically and experimentally, that the survival probability for an unstable quantum state, formed at $t = 0$, is not a simple exponential function, even if the latter is a good approximation for intermediate times. Typically, unstable quantum states/particles can decay in more than a single decay channel. In this work, the general expression for the probability that an unstable state decays into a certain i-th channel between the initial time $t = 0$ and an arbitrary $t > 0$ is provided, both for nonrelativistic quantum states and for relativistic particles. These partial decay probabilities are also not expone
be in the exponential limit. Qui
them potentially interesting in a
unexplored framework to search
systems, such as quantum tunneli

Found Phys (2012) 42:1262–1299
DOI 10.1007/s10701-012-9667-3

Non-exponential Decay in Quantum Field Theory and in Quantum Mechanics: The Case of Two (or More) Decay Channels

Francesco Giacosa

Exponential decay law

- N_0 : Number of unstable particles at the time t = 0.

$$N(t) = N_0 e^{-\Gamma t} \quad , \quad \tau = 1/\Gamma \text{ mean lifetime}$$

Confirmend in countless cases!

- For a single unstable particle:

$$p(t) = e^{-\Gamma t}$$

is the survival probability for a single unstable particle created at t=0.
(Intrinsic probabilty, see Schrödinger´s cat).

For small times: $p(t) = 1 - \Gamma t + ...$

Basic definitions

Uniwersytet Jana Kochanowskiego w Kielcach

Let $|S\rangle$ be an unstable state prepared at t = 0 .

Survival probabilty ammplitude at t > 0 :

$$a(t) = \langle S | e^{-iHt} | S \rangle$$

Survival probability : $p(t) = |a(t)|^2$

Rep. Prog. Phys., Vol. 41, 1978. Printed in Great Britain

Decay theory of unstable quantum systems

L FONDA, G C GHIRARDI and A RIMINI

Deviations from the exp. law at short times

Taylor expansion of the amplitude:

$$a(t) = \langle S | e^{-iHt} | S \rangle = 1 - it \langle S | H | S \rangle - \frac{t^2}{2} \langle S | H^2 | S \rangle + \ldots$$

$$a^*(t) = \langle S | e^{-iHt} | S \rangle = 1 + it \langle S | H | S \rangle - \frac{t^2}{2} \langle S | H^2 | S \rangle + \ldots$$

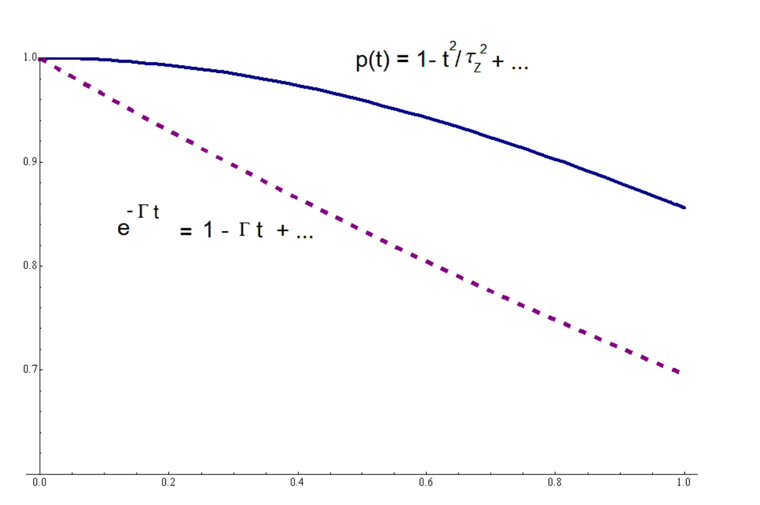
It follows:

$$\mathrm{p(t)} = |\mathrm{a(t)}|^2 = \mathrm{a}^*(\mathrm{t})\mathrm{a}(\mathrm{t}) = 1 - \mathrm{t}^2 \left(\langle \mathrm{S} | \mathrm{H}^2 | \mathrm{S} \rangle - \langle \mathrm{S} | \mathrm{H} | \mathrm{S} \rangle^2 \right) + \ldots = 1 - \frac{\mathrm{t}^2}{\tau_Z^2} + \ldots$$

p(t) decreases quadratically (not linearly); no exp. decay for short times. τ_Z is the `Zeno time´.

where $\tau_Z = \dfrac{1}{\sqrt{\langle \mathrm{S} | \mathrm{H}^2 | \mathrm{S} \rangle - \langle \mathrm{S} | \mathrm{H} | \mathrm{S} \rangle^2}}$.

Note: the quadratic behavior holds
for any quantum transition, not only for decays.



Time evolution and energy distribution (1)

Uniwersytet Jana Kochanowskiego w Kielcach

The unstable state $|S\rangle$ is not an eigenstate of the Hamiltonian H.

Let $d_S(E)$ be the energy distribution of the unstable state $|S\rangle$.

Normalization holds: $\int_{-\infty}^{+\infty} d_S(E) dE = 1$

$$p(t) = \left| \int_{E_{th,1}}^{\infty} \mathrm{dE} d_S(E) e^{-\frac{i}{\hbar}Et} \right|^2$$

In stable limit : $d_S(E) = \delta(E - M) \rightarrow a(t) = e^{-iMt} \rightarrow p(t) = 1$

Time evolution and energy distribution (2)

Breit-Wigner distribution:

$$d_S(E) = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \Gamma^2/4} \rightarrow a(t) = e^{-iM_0 t - \Gamma t/2} \rightarrow p(t) = e^{-\Gamma t}.$$

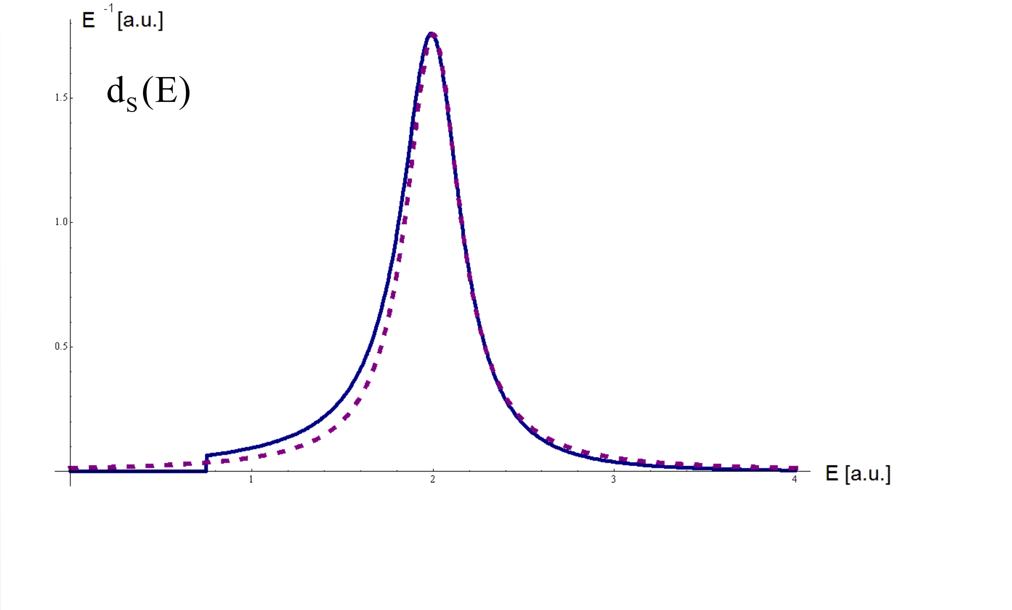
The Breit-Wigner energy distribution cannot be exact.

Two physical conditions for a realistic $d_S(E)$ are:

1) Minimal energy: $d_S(E) = 0 \text{ for } E < E_{\min}$

2) Mean energy finite: $\langle \mathrm{E} \rangle = \int_{-\infty}^{+\infty} d_S(E) E dE = \int_{E_{\min}}^{+\infty} d_S(E) E dE < \infty$

A very simple numerical example

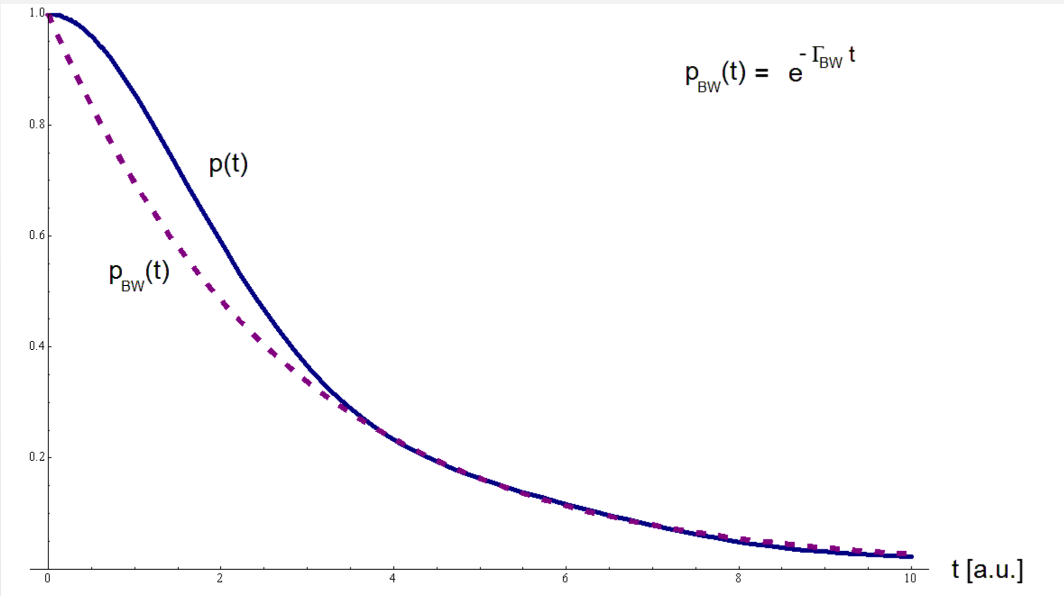


$$M_0 = 2;\ E_{\min} = 0.75;\ \ \Gamma = 0.4;\ \Lambda = 3$$

$$d_S(E) = N_0 \frac{\Gamma}{2\pi} \frac{e^{-(E^2-E_0^2)/\Lambda^2}\,\theta(E-E_{\min})}{(E-M_0)^2+\Gamma^2/4}$$

$$d_{BW}(E) = \frac{\Gamma_{BW}}{2\pi} \frac{1}{(E-M_0)^2+\Gamma_{BW}{}^2/4}$$

Γ_{BW}, such that $d_{BW}(M_0) = d_S(M_0)$



$$a(t) = \int_{-\infty}^{+\infty} d_S(E) e^{-iEt} dE\ ; \quad p(t) = |a(t)|^2$$

$$p_{BW}(t) = e^{-\Gamma_{BW} t}$$

Experimental confirmation of non-exponential decay (1)

NATURE | VOL 387 | 5 JUNE 1997

Experimental evidence for non-exponential decay in quantum tunnelling

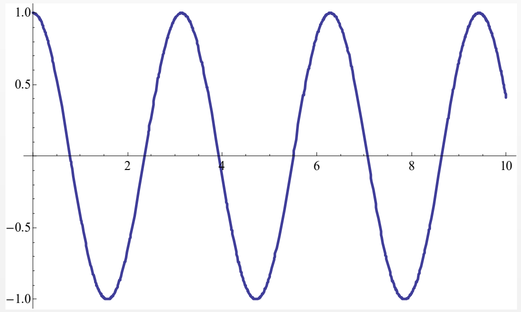
Steven R. Wilkinson, Cyrus F. Bharucha, Martin C. Fischer, Kirk W. Madison, Patrick R. Morrow, Qian Niu, Bala Sundaram* & Mark G. Raizen

Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1081, USA

An exponential decay law is the universal hallmark of unstable systems and is observed in all fields of science. This law is not, however, fully consistent with quantum mechanics and deviations from exponential decay have been predicted for short as well as long times[1–8]. Such deviations have not hitherto been observed experimentally. Here we present experimental evidence for short-time deviation from exponential decay in a quantum tunnelling experiment. Our system consists of ultra-cold sodium atoms that are trapped in an accelerating periodic optical potential created by a standing wave of light. Atoms can escape the wells by quantum tunnelling, and the number that remain can be measured as a function of interaction time for a fixed value of the well depth and acceleration. We observe that for short times the survival probability is initially constant before developing the characteristics of exponential decay. The conceptual simplicity of the experiment enables a detailed comparison with theoretical predictions.

Cold Na atoms in a optical potential

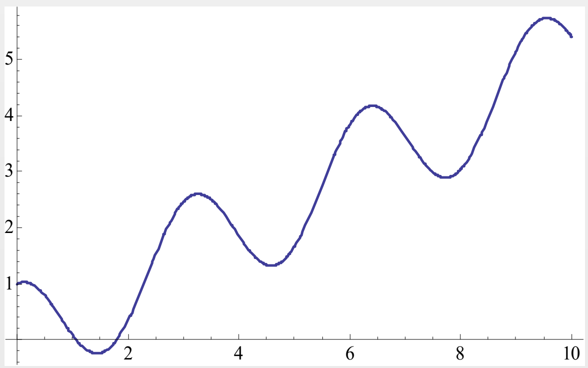
$V(x,0)$



x [a.u.]

$$V(x,t) = V_0 \cos\left(2k_L x - k_L a t^2\right)$$

$U(x')$



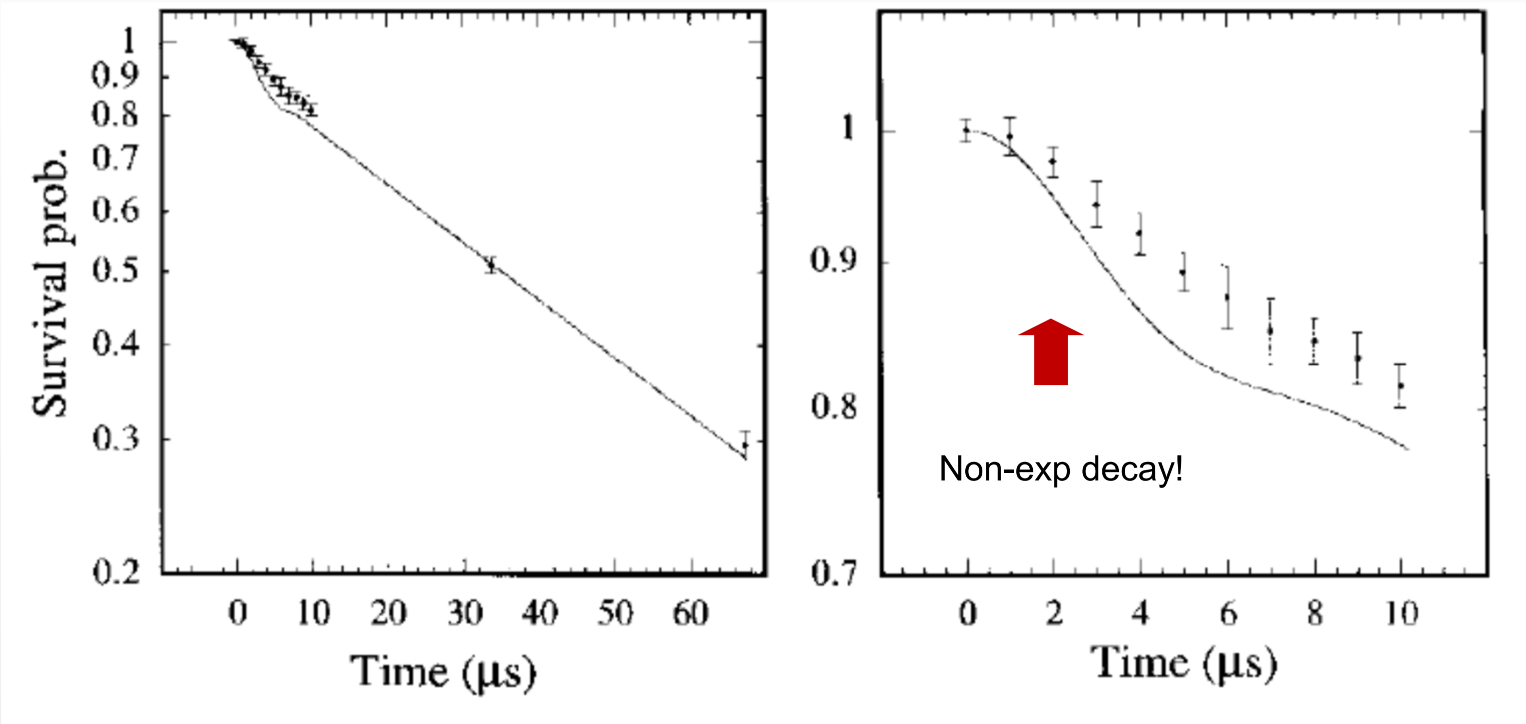
x' [a.u.]

$$x' = x - \frac{1}{2} a t^2$$

$$U(x') = V_0 \cos\left(2k_L x'\right) + Max'$$

Experimental confirmation of non-exponential decay (2)

Measured survival probabilty p(t)



Experimental confirmation of non-exponential decay and Zeno /Anti-Zeno effects

VOLUME 87, NUMBER 4 PHYSICAL REVIEW LETTERS 23 JULY 2001

Observation of the Quantum Zeno and Anti-Zeno Effects in an Unstable System

M. C. Fischer, B. Gutiérrez-Medina, and M. G. Raizen
Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1081
(Received 30 March 2001; published 10 July 2001)

We report the first observation of the quantum Zeno and anti-Zeno effects in an unstable system. Cold sodium atoms are trapped in a far-detuned standing wave of light that is accelerated for a controlled duration. For a large acceleration the atoms can escape the trapping potential via tunneling. Initially the number of trapped atoms shows strong nonexponential decay features, evolving into the characteristic exponential decay behavior. We repeatedly measure the number of atoms remaining trapped during the initial period of nonexponential decay. Depending on the frequency of measurements we observe a decay that is suppressed or enhanced as compared to the unperturbed system.

Same exp. setup,
but with measurements in between

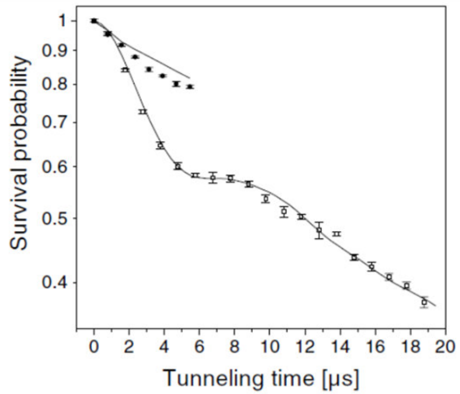


FIG. 3. Probability of survival in the accelerated potential as a function of duration of the tunneling acceleration. The hollow squares show the noninterrupted sequence, and the solid circles show the sequence with interruptions of 50 μs duration every 1 μs. The error bars denote the error of the mean. The data have been normalized to unity at $t_{\text{tunnel}} = 0$ in order to compare with the simulations. The solid lines are quantum mechanical simulations of the experimental sequence with no adjustable parameters. For these data the parameters were $a_{\text{tunnel}} = 15\,000$ m/s^2, $a_{\text{interr}} = 2000$ m/s^2, $t_{\text{interr}} = 50$ μs, and $V_0/h = 91$ kHz, where h is Planck's constant.

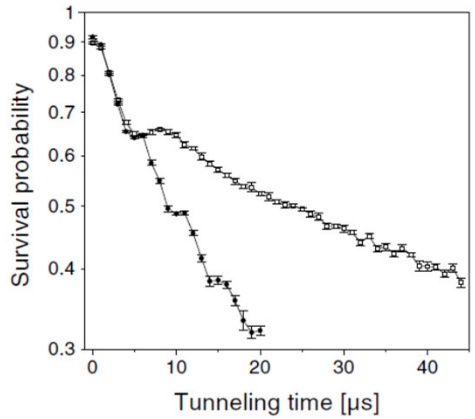


FIG. 4. Survival probability as a function of duration of the tunneling acceleration. The hollow squares show the noninterrupted sequence, and the solid circles show the sequence with interruptions of 40 μs duration every 5 μs. The error bars denote the error of the mean. The experimental data points have been connected by solid lines for clarity. For these data the parameters were: $a_{\text{tunnel}} = 15\,000$ m/s^2, $a_{\text{interr}} = 2800$ m/s^2, $t_{\text{interr}} = 40$ μs, and $V_0/h = 116$ kHz.

Zeno effekt

Anti-Zeno effect

Late-time deviations

PRL **96,** 163601 (2006) PHYSICAL REVIEW LETTERS week ending 28 APRIL 2006

Violation of the Exponential-Decay Law at Long Times

C. Rothe, S. I. Hintschich, and A. P. Monkman

Department of Physics, University of Durham, Durham, DH1 3LE, United Kingdom

(Received 4 July 2005; published 26 April 2006)

First-principles quantum mechanical calculations show that the exponential-decay law for any metastable state is only an approximation and predict an asymptotically algebraic contribution to the decay for sufficiently long times. In this Letter, we measure the luminescence decays of many dissolved organic materials after pulsed laser excitation over more than 20 lifetimes and obtain the first experimental proof of the turnover into the nonexponential decay regime. As theoretically expected, the strength of the nonexponential contributions scales with the energetic width of the excited state density distribution whereas the slope indicates the broadening mechanism.

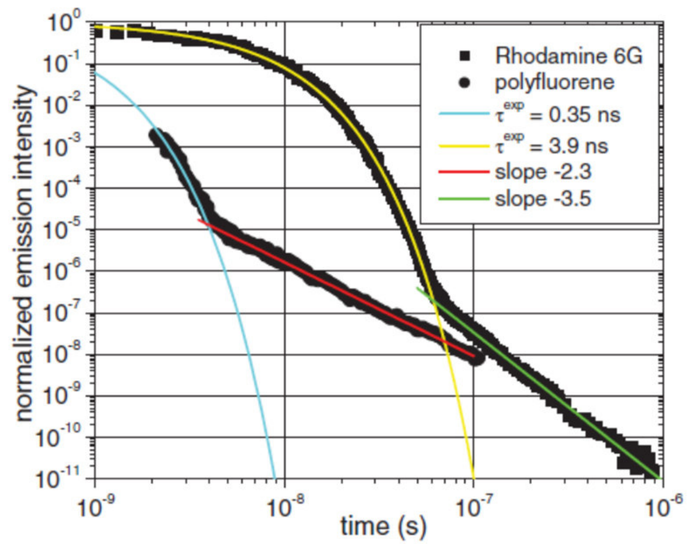


FIG. 2 (color). Corresponding double logarithmic fluorescence decays of the emissions shown in Fig. 1. Exponential and power law regions are indicated by solid lines and the emission intensity at time zero has been normalized.

Confirmation of: L. A. Khalfin. 1957. 1957 (Engl. trans. Zh.Eksp.Teor.Fiz.,33,1371)

Considerations

- No other short- or long-time deviation from the exp. law was seen in unstable states.
- Verification of the two aforementioned works (Reizen + Rothe) would be needed.
- The measurement of deviations in simple natural systems (elementary particles, nuclei, atoms) would be a great achievement.

Probability of decay: formal argument

$$p(t) = \left| \int_{E_{th,1}}^{\infty} \mathrm{dE} d_S(E) e^{-\frac{i}{\hbar}Et} \right|^2$$

$$\Gamma \to \Gamma(E)$$

$$d_S(E) = -\frac{1}{\pi} \operatorname{Im} \left[\frac{1}{E - M + \Pi(E)} \right]$$

with $\Pi(E)$ being the 'loop' wihth

$$\operatorname{Im} \Pi(E) = \Gamma(E)/2$$

$\operatorname{Re} \Pi(E)$ can be obtained by dispersion relations. (In some cases, one may directly calculate $\Pi(E)$).

A simple question

Next, a simple question might be asked: Which is the probability that the decay of the unstable state occurs between $t = 0$ and the time t? The answer is trivial, since the probability that the decay has actually occurred, denoted as $w(t)$, must be

$$w(t) = 1 - p(t)$$

Similarly, the quantity $h(t) = w'(t) = -p'(t)$ is the probability decay density, with $h(t)dt$ being the probability that the decay occurs between t and $t + dt$.

$$h(t) = w'(t) = -p'(t)$$

A difficult question

Then, a natural, but less easy question is the following: *How to calculate, in a general fashion, the probability, denoted as $w_i(t)$, that the decay occurs in the i-th channel between 0 and t?*

Which are the $w_i(t)$????

N Decay channels: formal aspects

$$\Gamma_i(E)$$

$i = 1, \ldots, N$ enumerates the decay channels

$$E_{th,1} \leq E_{th,2} \leq \ldots \leq E_{th,N}$$

$$\Pi(E) = \sum_{i=1}^{N} \Pi_i(E) \, , \Gamma_i(E) = 2 \, \mathrm{Im} \, \Pi_i(E)$$

partial BW widths are $\Gamma_i = \Gamma_i(M)$

$$\Gamma = \Gamma(M) = \sum_{i=1}^{N} \Gamma_i$$

Solution

$$w_i(t) = \int_{E_{th,i}}^{\infty} \mathrm{dE} \frac{\Gamma_i(E)}{2\pi} \left| \int_{E_{th,1}}^{+\infty} \mathrm{dE}' d_S(E') \frac{e^{-\frac{i}{\hbar}E't} - e^{-\frac{i}{\hbar}Et}}{E' - E} \right|^2$$

$$h_i(t) = w_i'(t)$$

$h_i(t)dt$ is the probability that the unstable particle decays in the i-th channel between t and t+dt

Breit-Wigner limit

$$d_S(E) = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \Gamma^2/4}$$

$$\Gamma = \sum_{i=1}^{N} \Gamma_i$$

$$p(t) = e^{-\frac{\Gamma}{\hbar}t}$$

$$w_i(t) = \frac{\Gamma_i}{\Gamma} w(t) = \frac{\Gamma_i}{\Gamma}\left(1 - e^{-\frac{\Gamma}{\hbar}t}\right)$$

$$h_i(t) = \frac{\Gamma_i}{\Gamma} h(t) = \frac{\Gamma_i}{\Gamma} e^{-\frac{\Gamma}{\hbar}t}.$$

$$\frac{w_i(t)}{w_j(t)} = \frac{h_i(t)}{h_j(t)} = \frac{\Gamma_i}{\Gamma_j} = const.$$

A simple model $\Gamma_i(E) = 2g_i^2 \frac{\sqrt{E - E_{th,i}}}{E^2 + \Lambda^2}$

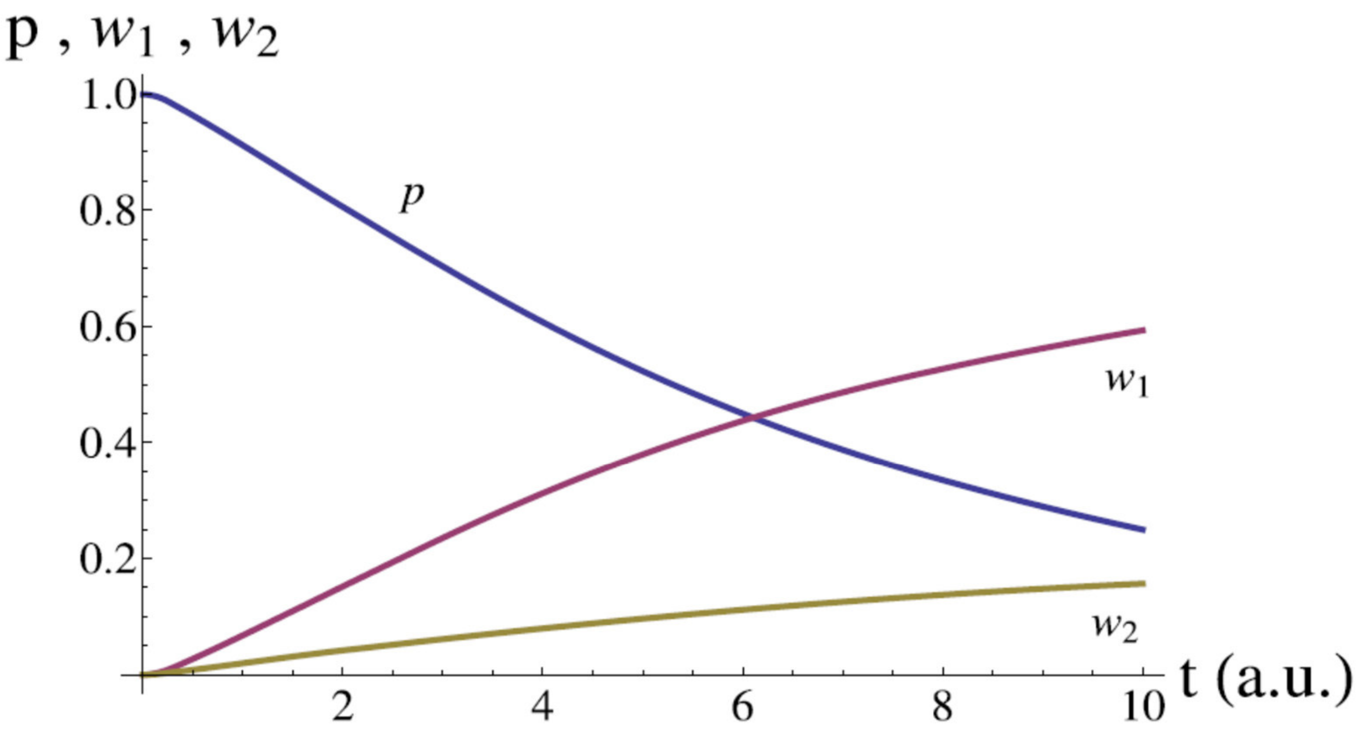


Fig. 1. The survival probability $p(t)$ of Eq. (1) and the decay probabilities $w_1(t)$ and $w_2(t)$ of Eq. (14) are plotted as function of t. The constraint $p + w_1 + w_2 = 1$ holds. Note, t is expressed in a.u. of $[M^{-}1]$.

$g_1/\sqrt{M} = 1,\ g_2/\sqrt{M} = 0.6$ $\quad E_{th,1}/M = 1/10,\ E_{th,2}/M = 1/2,\ \Lambda/M = 4,$

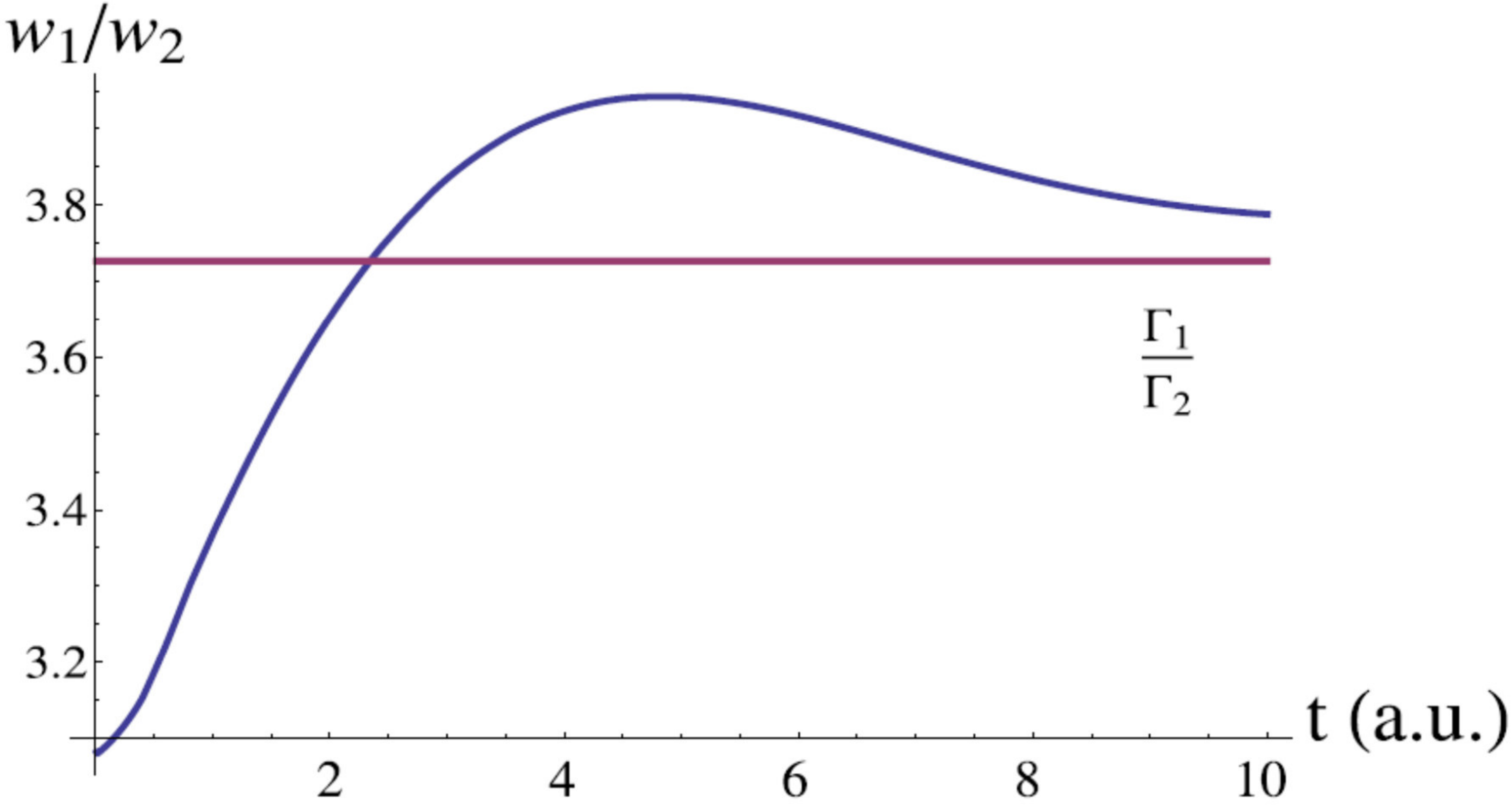


Fig. 2. The ratio w_1/w_2 is plotted as function of t. The straight line corresponds to the BW limit Γ_1/Γ_2, see Eq. (19).

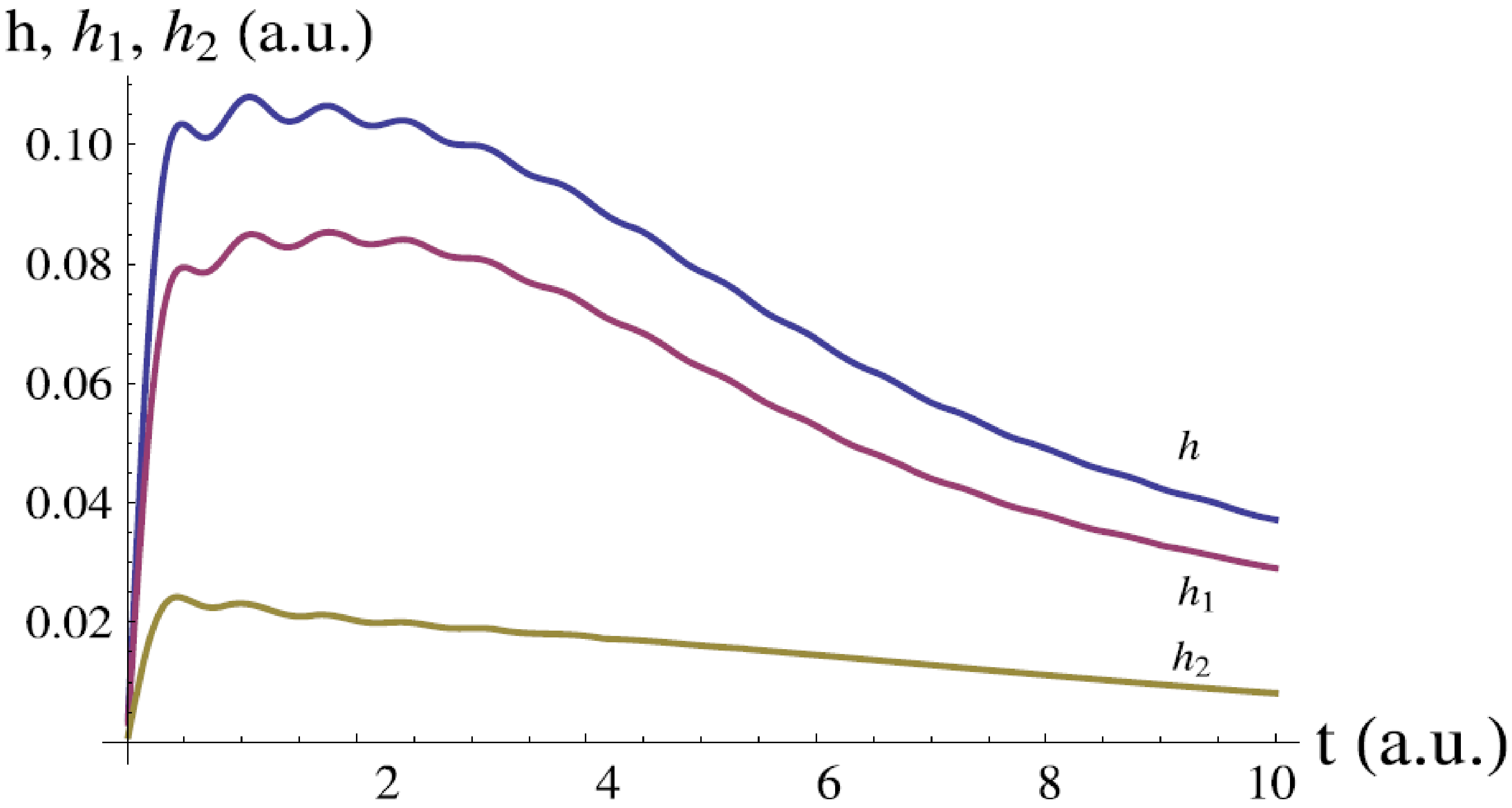


Fig. 3. The quantity $h(t) = w'(t) = -p'(t)$ as well as $h_i(t) = w_i'(t)$ is plotted. The equality $h(t) = h_1(t) + h_2(t)$ holds. Note, h and h_i are in units of $[M]$.

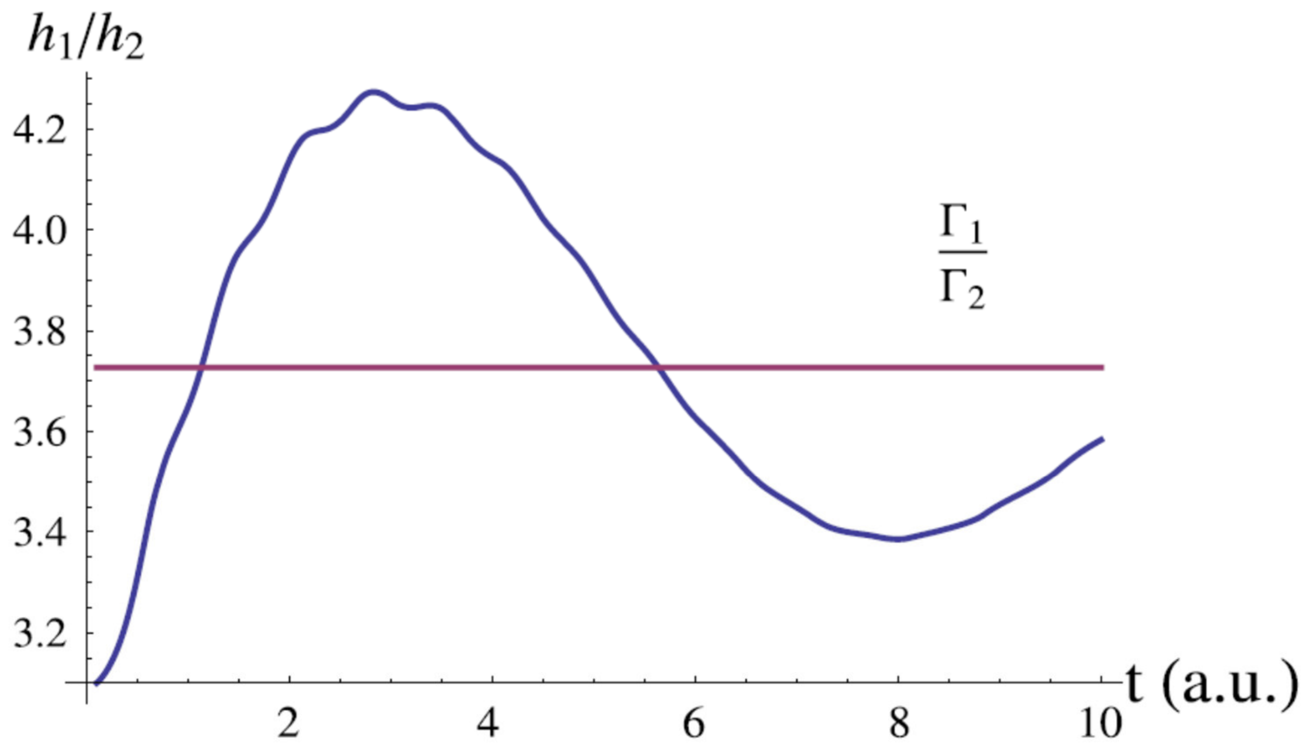
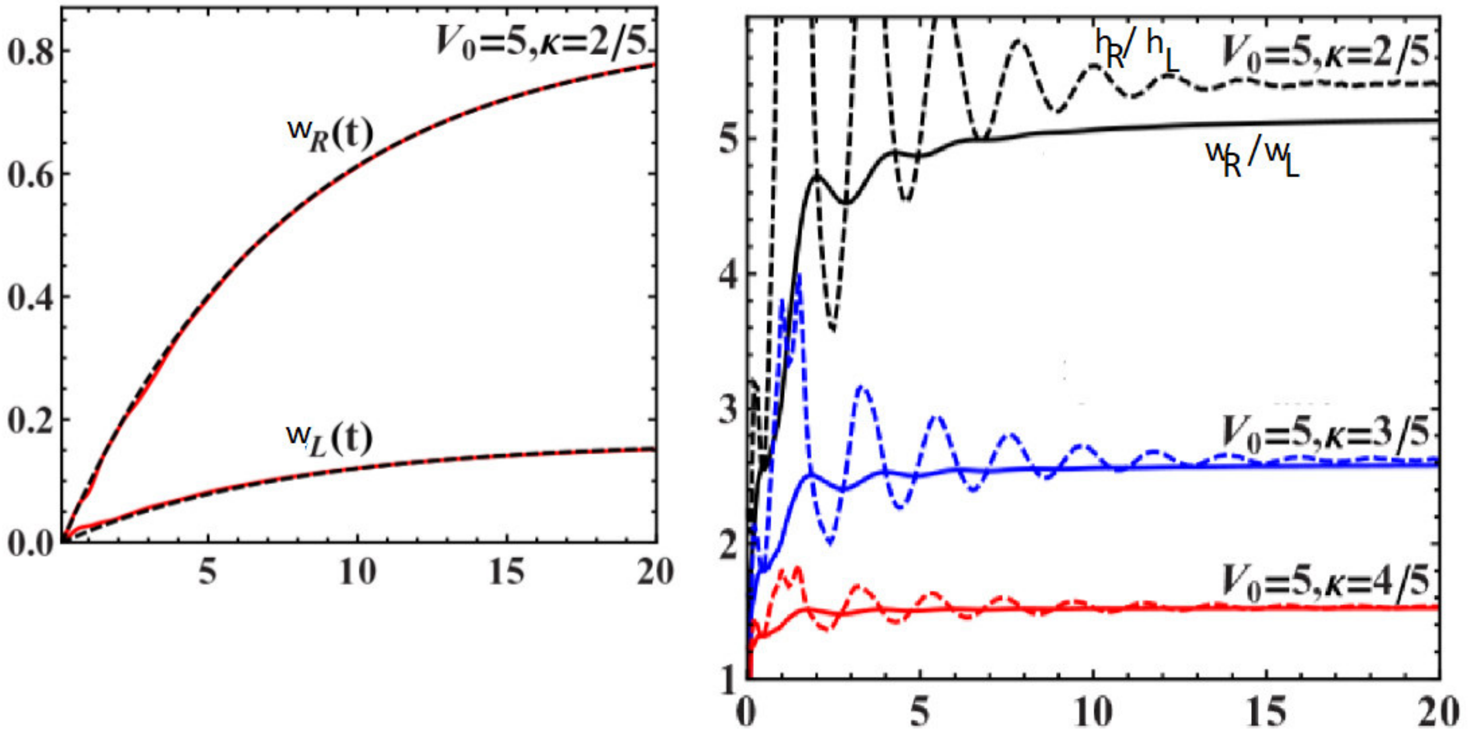


Fig. 4. Ratio h_1/h_2 as function of t. The straight line corresponds to the BW limit Γ_1/Γ_2, see Eq. (19). For the time intervals where $h_1/h_2 > \Gamma_1/\Gamma_2$, the decay in the first channel is enhanced (the opposite is true for $h_1/h_2 < \Gamma_1/\Gamma_2$).

Double-delta: $V_0[\delta(x+1) + \kappa\delta(x-1)]$



PHYSICAL REVIEW A **102**, 022204 (2020)

Capturing nonexponential dynamics in the presence of two decay channels

Francesco Giacosa,[1,2] Przemysław Kościk,[3] and Tomasz Sowiński[4]

[1]*Institute of Physics, Jan-Kochanowski University, ulica Uniwersytecka 7, PL-25406 Kielce, Poland*

[2]*Institute for Theoretical Physics, J. W. Goethe University, Max-von-Laue-Strasse 1, DE-60438 Frankfurt am Main, Germany*

[3]*Department of Computer Science, State Higher Vocational School in Tarnów, ulica Mickiewicza 8, PL-33100 Tarnów, Poland*

[4]*Institute of Physics, Polish Academy of Sciences, Aleja Lotników 32/46, PL-02668 Warsaw, Poland*

QFT case: result

$$w_i(t) = \int_{E_{th,i}}^{\infty} \mathrm{dE} \frac{2E^2\Gamma_i(E)}{\pi} \left| \int_{E_{th,1}}^{\infty} \mathrm{dE}' d_S(E') \frac{e^{-iE't} - e^{-iEt}}{E'^2 - E^2} \right|^2$$

Recalll QM:

$$w_i(t) = \int_{E_{th,i}}^{\infty} \mathrm{dE} \frac{\Gamma_i(E)}{2\pi} \left| \int_{E_{th,1}}^{+\infty} \mathrm{dE}' d_S(E') \frac{e^{-\frac{i}{\hbar}E't} - e^{-\frac{i}{\hbar}Et}}{E' - E} \right|^2$$

QFT: example

Uniwersytet Jana Kochanowskiego w Kielcach

$$\Gamma_i(s) = g_i^2 \sqrt{(s - s_{th,i})/s}$$

Eur. Phys. J. A (2021) 57:336
https://doi.org/10.1140/epja/s10050-021-00641-2

THE EUROPEAN PHYSICAL JOURNAL A

Check for updates

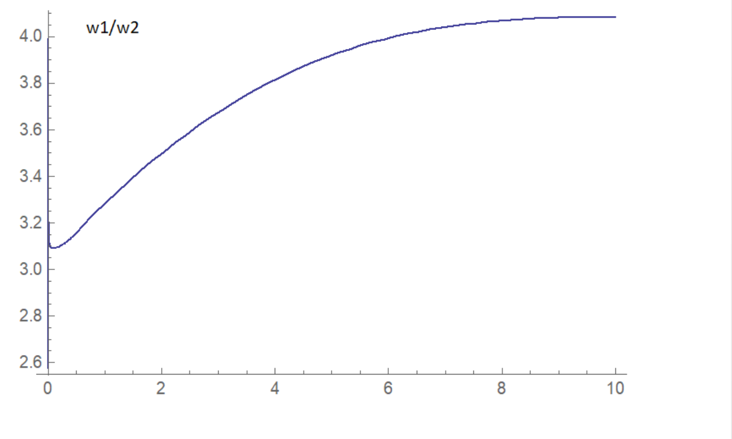
Regular Article - Theoretical Physics

A simple alternative to the relativistic Breit–Wigner distribution

Francesco Giacosa[1,2], **Anna Okopińska**[1], **Vanamali Shastry**[1,a]

[1] Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406 Kielce, Poland
[2] Institute for Theoretical Physics, Johann Wolfgang Goethe—University, Max von Laue-Str. 1, 60438 Frankfurt, Germany

$$d_S^{\mathrm{Sill}}(E) = \frac{2E}{\pi} \frac{\sqrt{E^2 - E_{th}^2}\,\tilde{\Gamma}}{(E^2 - M^2)^2 + (\sqrt{E^2 - E_{th}^2}\,\tilde{\Gamma})^2} \theta(E - E_{th})$$



Other decay-related topics

Uniwersytet Jana Kochanowskiego w Kielcach

PHYSICAL REVIEW A **90**, 052107 (2014)

Pulsed and continuous measurements of exponentially decaying systems

Francesco Giacosa[1,2] and Giuseppe Pagliara[3]

PHYSICAL REVIEW D **101,** 056003 (2020)

Measurement of the neutron lifetime and inverse quantum Zeno effect

Francesco Giacosa[1,2,*] and Giuseppe Pagliara[3,†]

PHYSICAL REVIEW A **104**, 052225 (2021)

Leggett-Garg inequalities and decays of unstable systems

Francesco Giacosa[1,2] and Giuseppe Pagliara[3,4]

Uniwersytet
Jana Kochanowskiego w Kielcach

Vol. 47 (2016) *ACTA PHYSICA POLONICA B* No 9

DECAY LAW AND TIME DILATATION*

Francesco Giacosa

Boosting Unstable Particles

L. Gavassino[1], and F. Giacosa[2,3]
[1]*Nicolaus Copernicus Astronomical Center, Polish Academy of Sciences, ul. Bartycka 18, 00-716 Warsaw, Poland*
[2]*Institute of Physics, Jan-Kochanowski Univ., ul. Uniwersytecka 7, 25-406 Kielce, Poland and*
[3]*Institute for Theoretical Physics, J. W. Goethe University,*
Max-von-Laue-Str. 1, 60438 Frankfurt, Germany.

2206.05125

Uniwersytet Jana Kochanowskiego w Kielcach

Thank You

Technical remark: Lee Hamiltonian

$$H_{\mathrm{L}} = M\,|S\rangle\langle S| + \sum_{i=1}^{N}\int_{E_{i,th}}^{\infty} \mathrm{d}E\,E\,|E,i\rangle\langle E,i|$$

$$+\sum_{i=1}^{N}\int_{E_{i,th}}^{\infty} \mathrm{d}E\sqrt{\frac{\Gamma_i(E)}{2\pi}}\left(|E,i\rangle\langle S| + h.c.\right)$$

$$w_i(t) = \int_{E_{th,i}}^{\infty} \mathrm{d}E\left|\left\langle E,i\left|e^{-\frac{i}{\hbar}H_{\mathrm{L}}t}\right|S\right\rangle\right|^2$$

Technical remark: the propagator

$$G_S(E) = \langle S| \frac{1}{E - H + i\varepsilon} |S\rangle = \frac{1}{E - M + \Pi(E) + i\varepsilon}$$

$$G_S(E) = \langle S| \frac{1}{E - H + i\varepsilon} |S\rangle = \frac{1}{E - M + \Pi(E) + i\varepsilon}$$

$$= \int_{E_{1,th}}^{+\infty} \mathrm{dE'} \frac{d_S(E')}{E - E' + i\varepsilon} \,.$$