Photon Emissions from Excited Hydrogen and Positronium

Francesco Giacosa
in collaboration with K. Kyzioł and M. Piotrowska

UJK Kielce (Poland) **& Goethe U Frankfurt** (Germany)

31/3/2025
UJ Krakow
Particle Physics Phenomenology and Experiments Seminar

Outline

- Part 1: Will we ever see deviation from the exp. decay law in elementary processes? A lesson from the electromagnetic decay of the 2P state of the H-atom
- Part 2: Can we apply hadronic techniques to the positronium? Do we learn something out of it?

part 1: H-atom

Basic definitions

Let $|\mathrm{S}\rangle$ be an unstable state prepared at $\mathrm{t} = 0$.

Survival probabilty ammplitude at $\mathrm{t} > 0$:

$$a(t) = \langle S | e^{-iHt} | S \rangle \qquad (\hbar = 1)$$

Survival probability: $p(t) = |a(t)|^2$

Rep. Prog. Phys., Vol. 41, 1978. Printed in Great Britain

Decay theory of unstable quantum systems

L FONDA, G C GHIRARDI and A RIMINI

Time evolution and energy distribution

The unstable state $|S\rangle$ is not an eigenstate of the Hamiltonian H.

Let $d_S(E)$ be the energy distribution of the unstable state $|S\rangle$.

Normalization holds: $\int_{-\infty}^{+\infty} d_S(E)dE = 1$

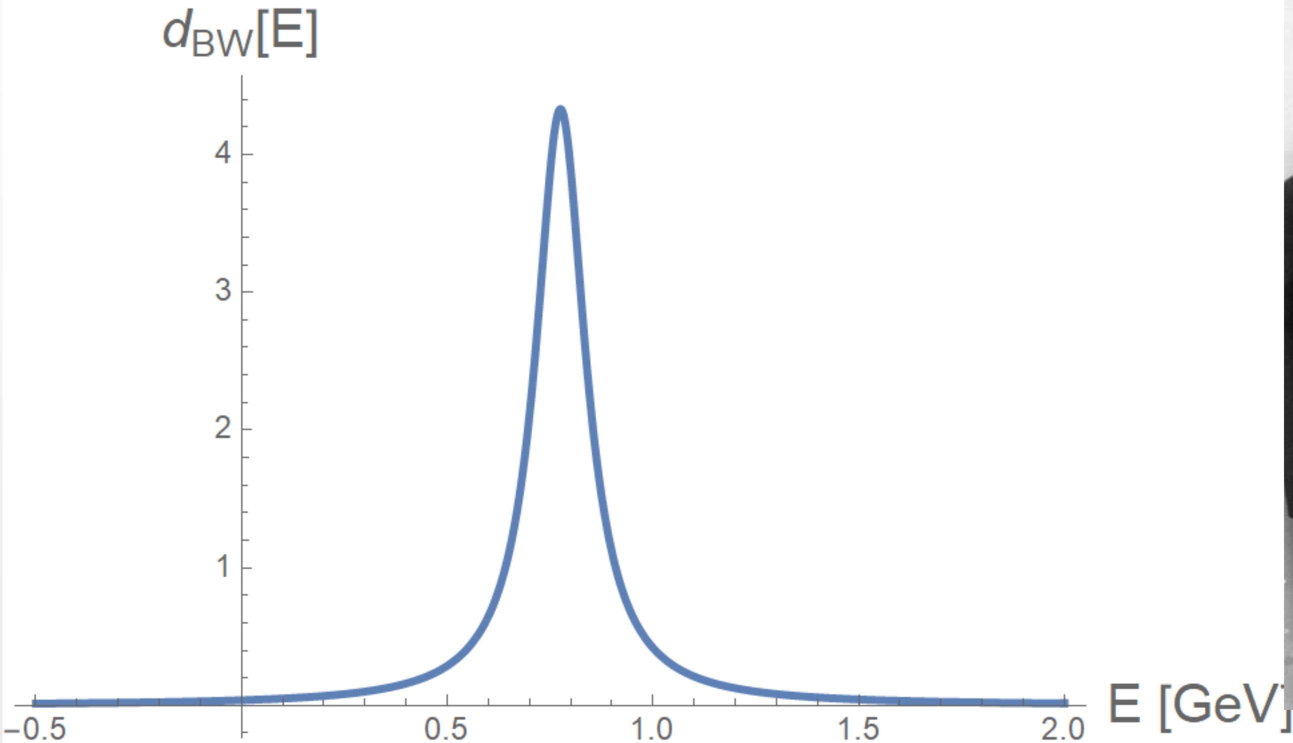
$$p(t) = \left| \int_{E_{th,1}}^{\infty} \mathrm{d}E\, d_S(E) e^{-\frac{i}{\hbar}Et} \right|^2$$

In stable limit: $d_S(E) = \delta(E-M) \rightarrow a(t) = e^{-iMt} \rightarrow p(t) = 1$

Payley and Wiener (1934) theorem: P(t) is not exponential at large time if a left-threshold is present

Breit-Wigner distribution

$$d_S^{\mathrm{BW}}(E) = \frac{\Gamma}{2\pi} \frac{1}{(E - M)^2 + \frac{\Gamma^2}{4}}$$



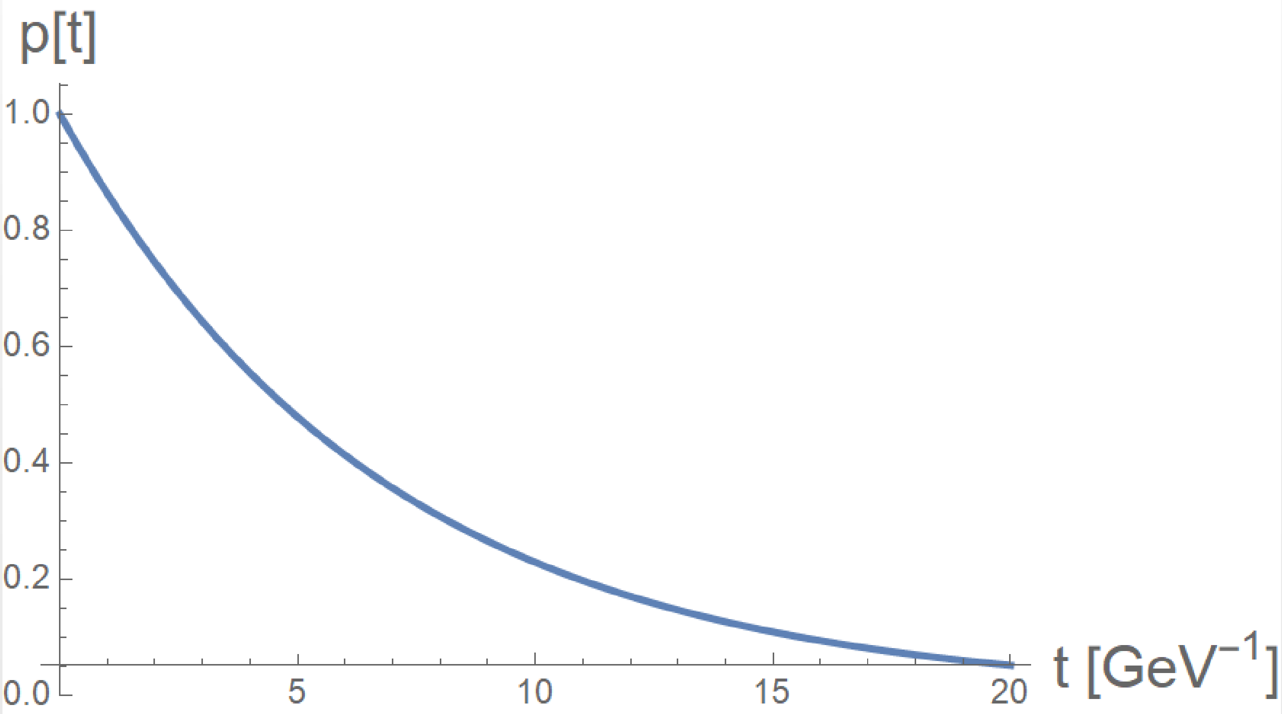
Rho-meson as example.

BW extends from –inf to +inf. There is no left threshold.

BW corresponds to exp. decay

$$a_S^{\mathrm{BW}}(t) = \int_{E_{th}}^{+\infty} \mathrm{dE} d_S^{\mathrm{BW}}(E) = e^{-iMt-\Gamma t/2}$$

$$p^{\mathrm{BW}}(t) = \left|a_S^{\mathrm{BW}}(t)\right|^2 = e^{-\Gamma t}$$



Going beyond Breit-Wigner:

We have seen that via the Breit-Wigner distribution:

$$d_S(E) = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \Gamma^2/4} \rightarrow a(t) = e^{-iM_0 t - \Gamma t/2} \rightarrow p(t) = e^{-\Gamma t}.$$

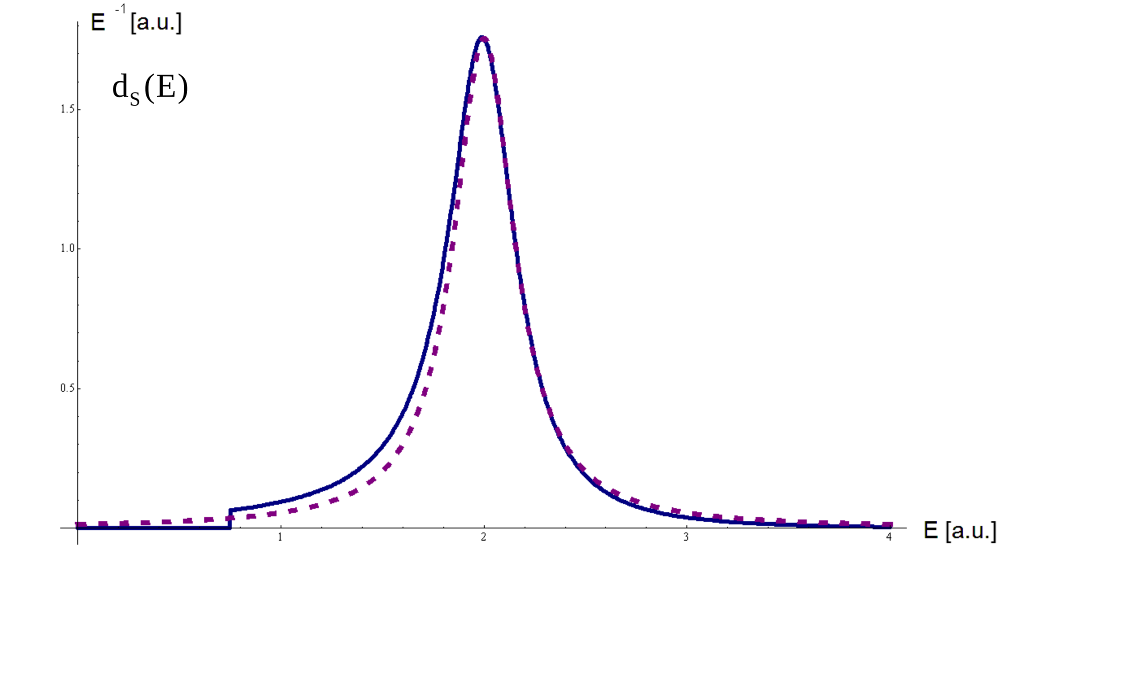
The Breit-Wigner energy distribution cannot be exact.

Two physical conditions for a realistic $d_S(E)$ are:

1) Minimal energy: $d_S(E) = 0 \text{ for } E < E_{\min}$

2) Mean energy finite: $\langle \mathrm{E} \rangle = \int_{-\infty}^{+\infty} d_S(E) E dE = \int_{E_{\min}}^{+\infty} d_S(E) E dE < \infty$

Example: threshold plus form factor



$$d_S(E) = N_0 \frac{\Gamma}{2\pi} \frac{e^{-(E^2 - E_0^2)/\Lambda^2} \theta(E - E_{min})}{(E - M_0)^2 + \Gamma^2/4}$$

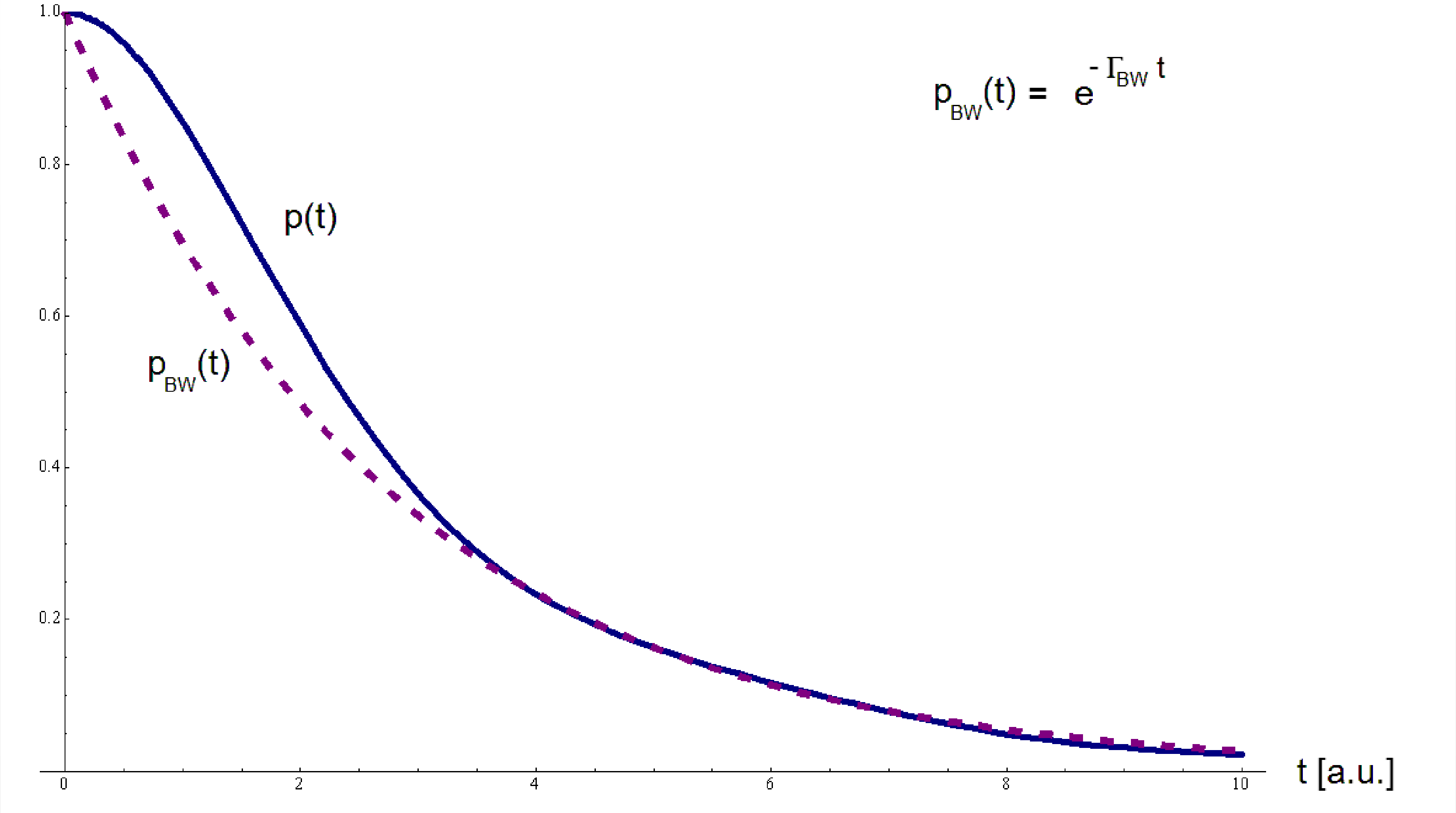
$$M_0 = 2;\ E_{min} = 0.75;\ \ \Gamma = 0.4; \Lambda = 3$$

$$a(t) = \int_{-\infty}^{+\infty} d_S(E) e^{-iEt} dE\,; \quad p(t) = |a(t)|^2$$

$$d_{BW}(E) = \frac{\Gamma_{BW}}{2\pi} \frac{1}{(E - M_0)^2 + \Gamma_{BW}{}^2/4}$$

$$p_{BW}(t) = e^{-\Gamma_{BW} t}$$

Γ_{BW}, such that $d_{BW}(M_0) = d_S(M_0)$



1.0
0.8
0.6
0.4
0.2
0
2
4
6
8
10
p(t)
$p_{BW}(t)$
$p_{BW}(t) = e^{-\Gamma_{BW}\, t}$
t [a.u.]

Experimental confirmation of non-exponential decay: short times

Cold Na atoms in a optical potential

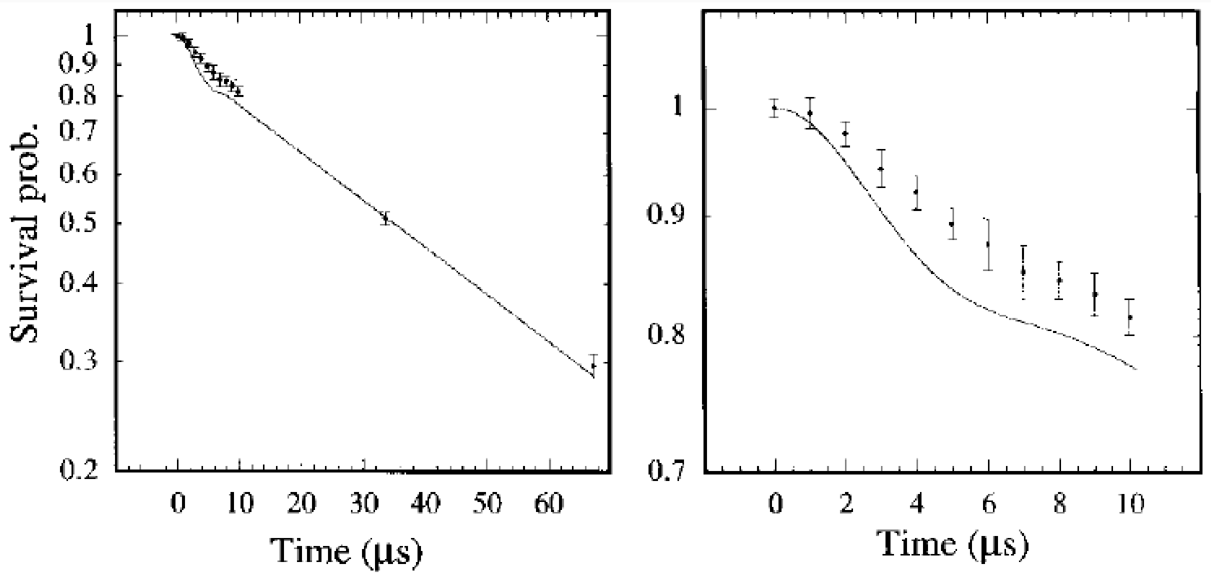
NATURE | VOL 387 | 5 JUNE 1997

Experimental evidence for non-exponential decay in quantum tunnelling

Steven R. Wilkinson, Cyrus F. Bharucha, Martin C. Fischer, Kirk W. Madison, Patrick R. Morrow, Qian Niu, Bala Sundaram* & Mark G. Raizen

Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1081, USA

An exponential decay law is the universal hallmark of unstable systems and is observed in all fields of science. This law is not, however, fully consistent with quantum mechanics and deviations from exponential decay have been predicted for short as well as long times[1–8]. Such deviations have not hitherto been observed experimentally. Here we present experimental evidence for short-time deviation from exponential decay in a quantum tunnelling experiment. Our system consists of ultra-cold sodium atoms that are trapped in an accelerating periodic optical potential created by a standing wave of light. Atoms can escape the wells by quantum tunnelling, and the number that remain can be measured as a function of interaction time for a fixed value of the well depth and acceleration. We observe that for short times the survival probability is initially constant before developing the characteristics of exponential decay. The conceptual simplicity of the experiment enables a detailed comparison with theoretical predictions.



Experimental confirmation of non-exponential decay: long times

PRL **96,** 163601 (2006) PHYSICAL REVIEW LETTERS week ending 28 APRIL 2006

Violation of the Exponential-Decay Law at Long Times

C. Rothe, S. I. Hintschich, and A. P. Monkman
Department of Physics, University of Durham, Durham, DH1 3LE, United Kingdom
(Received 4 July 2005; published 26 April 2006)

First-principles quantum mechanical calculations show that the exponential-decay law for any metastable state is only an approximation and predict an asymptotically algebraic contribution to the decay for sufficiently long times. In this Letter, we measure the luminescence decays of many dissolved organic materials after pulsed laser excitation over more than 20 lifetimes and obtain the first experimental proof of the turnover into the nonexponential decay regime. As theoretically expected, the strength of the nonexponential contributions scales with the energetic width of the excited state density distribution

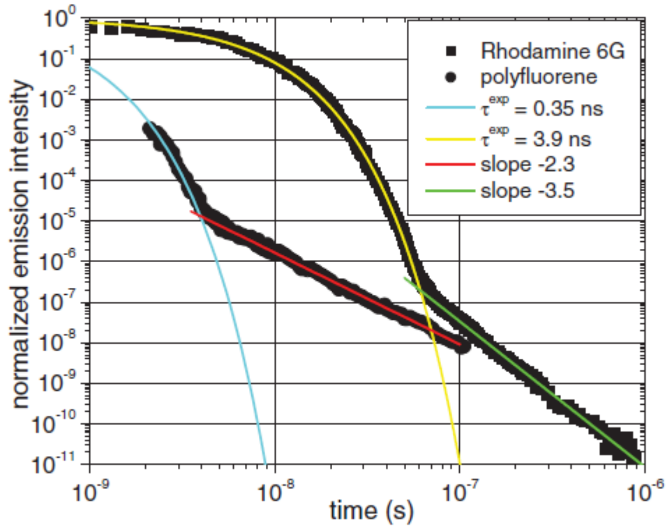


FIG. 2 (color). Corresponding double logarithmic fluorescence decays of the emissions shown in Fig. 1. Exponential and power law regions are indicated by solid lines and the emission intensity at time zero has been normalized.

Confirmation of: L. A. Khalfin. 1957. 1957 (Engl. trans. Zh.Eksp.Teor.Fiz.,33,1371)

Considerations/1

- No other short- or long-time deviation from the exp. law was ‘directly’seen in unstable states.

- Verification of the two aforementioned works (Reizen + Rothe) would be needed.

- The measurement of deviations in simple natural systems (elementary particles, nuclei, atoms) would be a great achievement.

Considerations/2

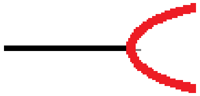
- The ‘brute force’ threshold and high-energy behavior can be good as a first approximation, but it is just an ‘ad hoc’ modification of Breit-Wigner.

- How to properly describe the theory of decay?

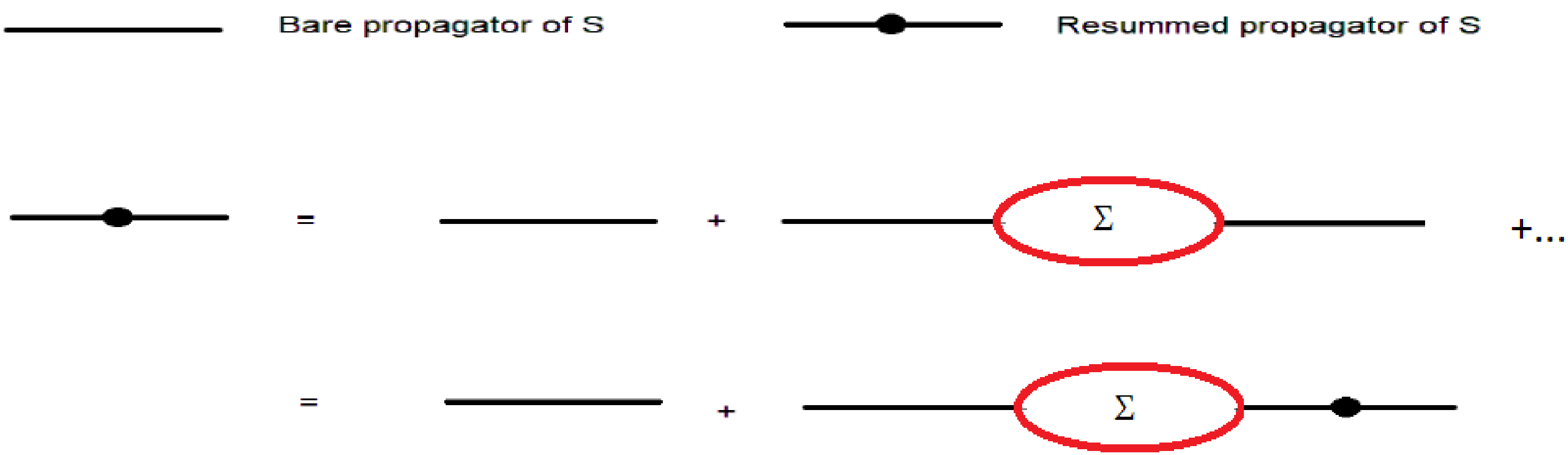
- Which is a suitable energy distribution for e.m. decays?

General Pictorial representation for un (unstable) state S

Decay



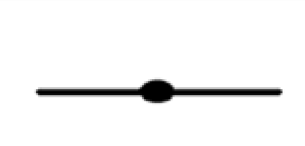
Propagator of S



General non-relativistic approach

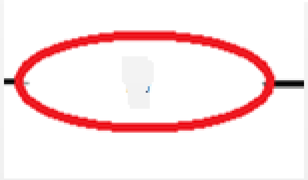
Full propagator

$$G_S(E) = \frac{1}{E - M + \Pi(E) + i\varepsilon}$$



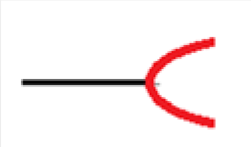
Self-energy (or loop)
E_{th} is the threshold energy

$$\Pi(E) = -\int_{E_{th}}^{\infty} \frac{1}{\pi} \frac{\mathrm{Im}\,\Pi(E')}{E - E' + i\varepsilon} dE'$$



Energy dependent ‘decay width’

$$\Gamma(E) = 2\,\mathrm{Im}\,\Pi(E)$$



Link between propagator and distribution

$$d_S(E) = -\frac{1}{\pi}\,\mathrm{Im}[G_S(E)] = \frac{1}{\pi}\frac{\mathrm{Im}\,\Pi(E)}{(E - M + \mathrm{Re}\,\Pi(E))^2 + (\mathrm{Im}\,\Pi(E))^2}$$

Normalization ok!!

$$\int_{E_{th}}^{\infty} \mathrm{dE}\, d_S(E) = 1$$

Symmetries in Science XVIII IOP Publishing
Journal of Physics: Conference Series **1612** (2020) 012012 doi:10.1088/1742-6596/1612/1/012012

The Lee model: a tool to study decays

Francesco Giacosa[a,b]
[a]Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406 Kielce, Poland

[arXiv:2001.07781 [hep-ph]]

Time-evolution (general)

$$a_S(t) = Ze^{-iz_{pole}t} + \ldots,$$

The dots describe short- and long-time deviations from the exponential decay

The pole:

$$z_{pole} - M + \Pi_{II}(z_{pole}) = 0 \,,$$

where II refers to the second Riemann sheet. Then:

$$z_{pole} = M_{pole} - i\frac{\Gamma_{pole}}{2} \,.$$

2P-1S transition of H-atom

ELSEVIER

27 April 1998

PHYSICS LETTERS A

Physics Letters A 241 (1998) 139–144

Temporal behavior and quantum Zeno time of an excited state of the hydrogen atom

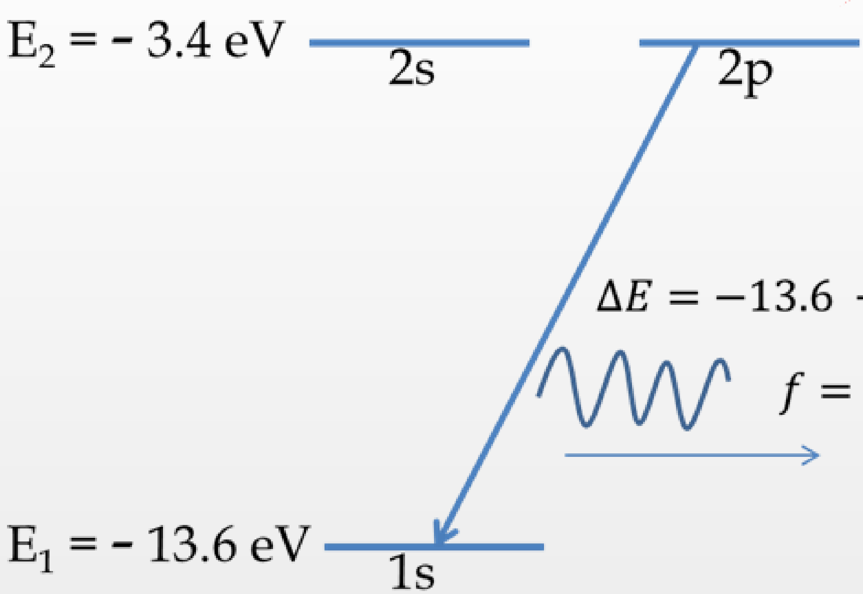
P. Facchi [1], S. Pascazio [2]

Dipartimento di Fisica, Università di Bari and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy

Received 4 November 1997; revised manuscript received 28 January 1998; accepted for publication 10 February 1998
Communicated by P.R. Holland

$$\mathrm{Im}[\Pi(E)] = \pi\chi\Lambda \frac{\frac{E-E_{th}}{\Lambda}}{\left(1+\left(\frac{E-E_{th}}{\Lambda}\right)^2\right)^4} \vartheta(E-E_{th}) \,,$$

$$\chi = \frac{2}{\pi}\left(\frac{2}{3}\right)^9 \alpha^3 \simeq 6.43509\times 10^{-9},\ \Lambda = \frac{3}{2}\alpha m_e \simeq 5593.41\,\mathrm{eV} \,.$$



$$\Delta E = -13.6 - (-3.4) = -10.2\ eV$$

$$f = \frac{\Delta E}{h} = \frac{10.2\ eV}{4.14\times10^{-15} eV.s} = 2.46 \times 10^{15} Hz$$

(UV light)

Decay width and lifetime of 2P level of H-atom

$$\Gamma = \frac{1}{\tau} = \frac{3}{2}\left(\frac{2}{3}\right)^9 \frac{m_e \alpha^5}{\left(1+\left(\frac{\alpha}{4}\right)^2\right)^4} = 4.12582 \times 10^{-7}\,\mathrm{eV}$$

$$\tau \simeq 2.42376 \times 10^6\,\mathrm{eV}^{-1} = 1.59535 \times 10^{-9}\,\mathrm{s}$$

ACTA PHYSICA POLONICA A No. 5 Vol. **146** (2024)

Proceedings of 5th Jagiellonian Symposium on Advances in Particle Physics and Medicine, Kraków, Poland, 2024

Nonexponential Decay Law of the $2P$–$1S$ Transition of the H Atom

F. Giacosa[a,b,*] and K. Kyzioł[a]

[a] *Institute of Physics, Jan Kochanowski University, Uniwersytecka 7, 25-406 Kielce, Poland*
[b] *Institute for Theoretical Physics, J.W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt, Germany*

Doi: 10.12693/APhysPolA.146.704 *e-mail: fgiacosa@ujk.edu.pl

Giacosa and K. Kyziol,
Nonexponential decay law of the 2P-1S
transition of the H-atom,
[arXiv:2408.06905 [quant-ph]].

Spectral function: analytic expression

$$\mathrm{Im}\left[\Pi(E)\right] = \frac{\gamma\sqrt{E-E_{th}}}{\left(1+\frac{(E-E_{th})^2}{\Lambda^2}\right)^2}$$

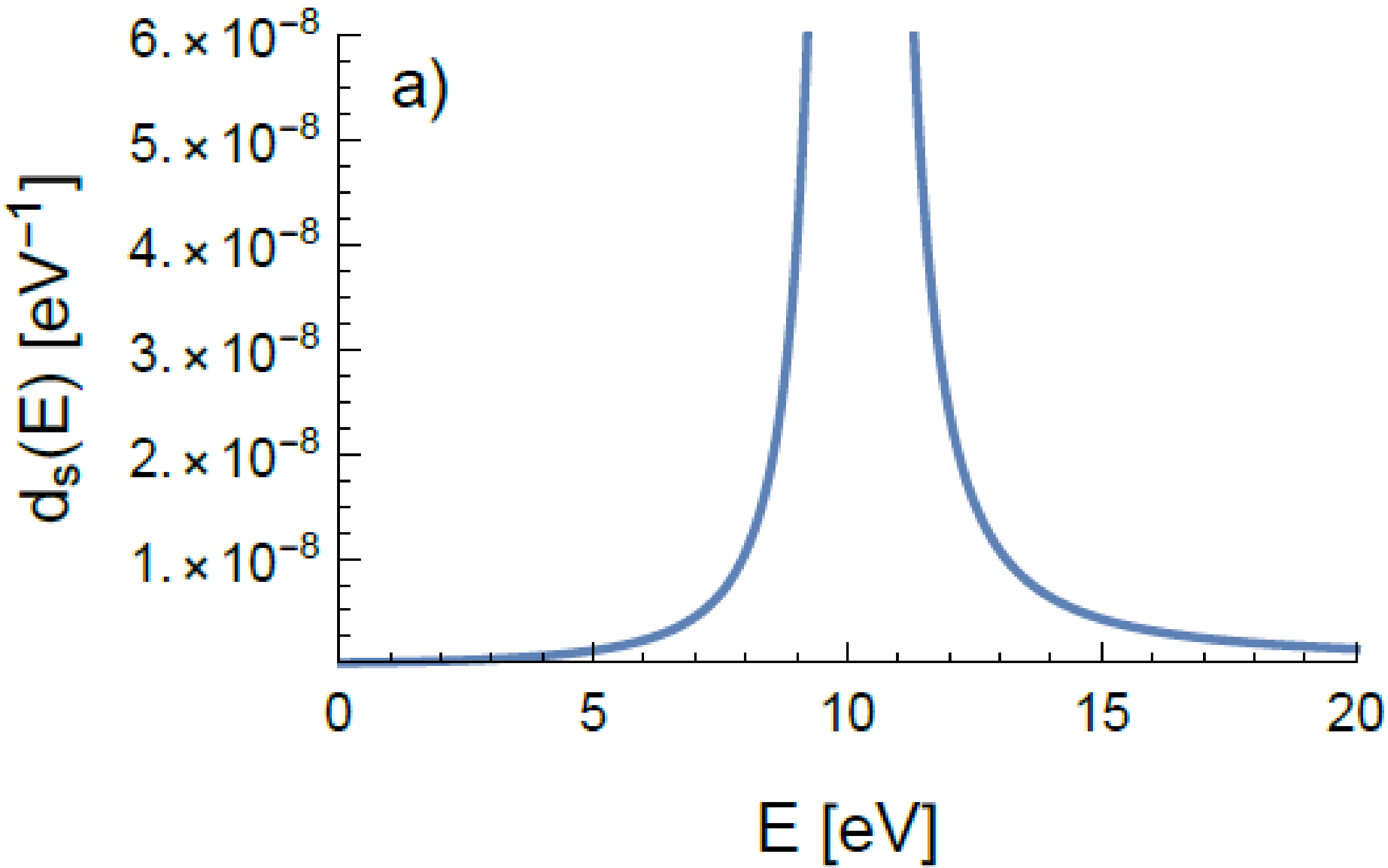
$$\frac{1}{\chi\Lambda}\mathrm{Re}[\Pi(E)] - C = -\frac{2\frac{E-E_{th}}{\Lambda}\ln\left(\frac{E-E_{th}}{\Lambda}\right)+\pi\left(\frac{E-E_{th}}{\Lambda}\right)^2}{2\left(1+\left(\frac{E-E_{th}}{\Lambda}\right)^2\right)^4} - $$
$$-\frac{2\frac{E-E_{th}}{\Lambda}+\pi\left(\frac{E-E_{th}}{\Lambda}\right)^2}{4\left(1+\left(\frac{E-E_{th}}{\Lambda}\right)^2\right)^3} - \frac{4\frac{E-E_{th}}{\Lambda}+3\pi\left(\frac{E-E_{th}}{\Lambda}\right)^2}{16\left(1+\left(\frac{E-E_{th}}{\Lambda}\right)^2\right)^2} + \frac{15\pi-16\frac{E-E_{th}}{\Lambda}}{96\left(1+\left(\frac{E-E_{th}}{\Lambda}\right)^2\right)}$$

2408.06905 [quant-ph]

$$d_S(E) = -\frac{1}{\pi}\mathrm{Im}[G_S(E)] = \frac{1}{\pi}\frac{\mathrm{Im}\,\Pi(E)}{(E-M+\mathrm{Re}\,\Pi(E))^2+(\mathrm{Im}\,\Pi(E))^2}$$

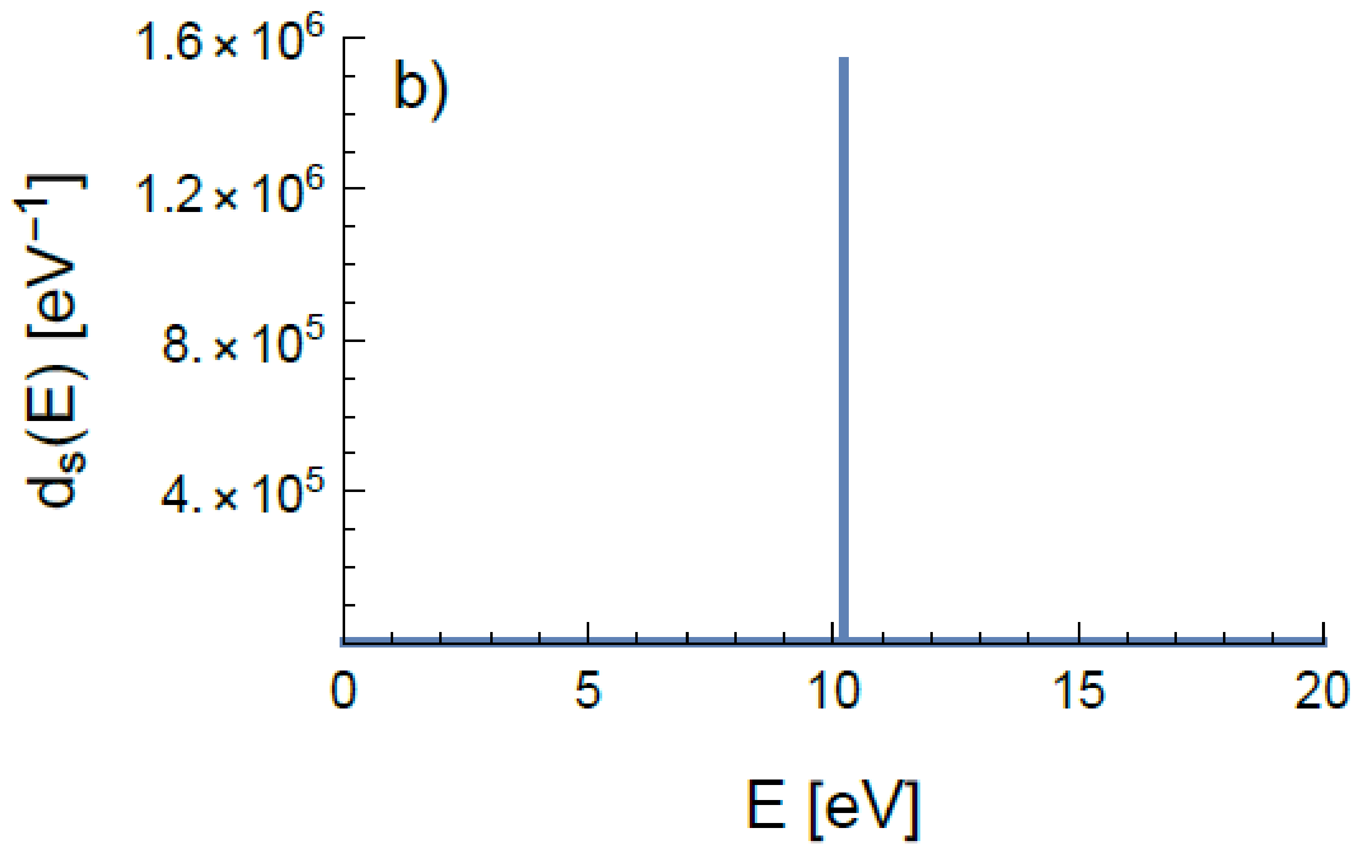
$$M = \frac{3}{8}\alpha^2 m_e \simeq 10.2043\,\mathrm{eV}$$

Spectral function: plot/1

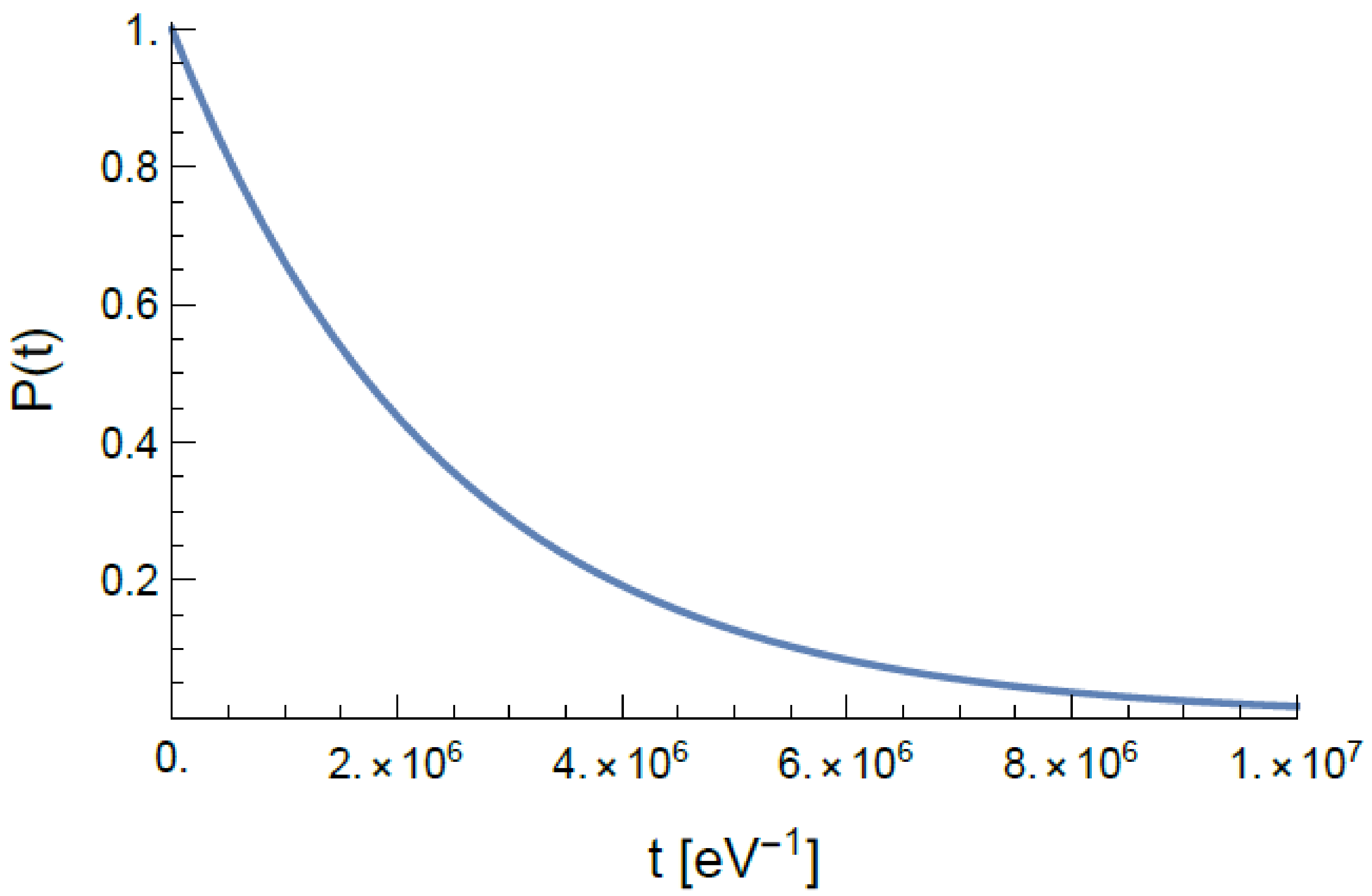


2408.06905 [quant-ph]

Spectral function/plot 2



Survival probability P(t)



2408.06905 [quant-ph]

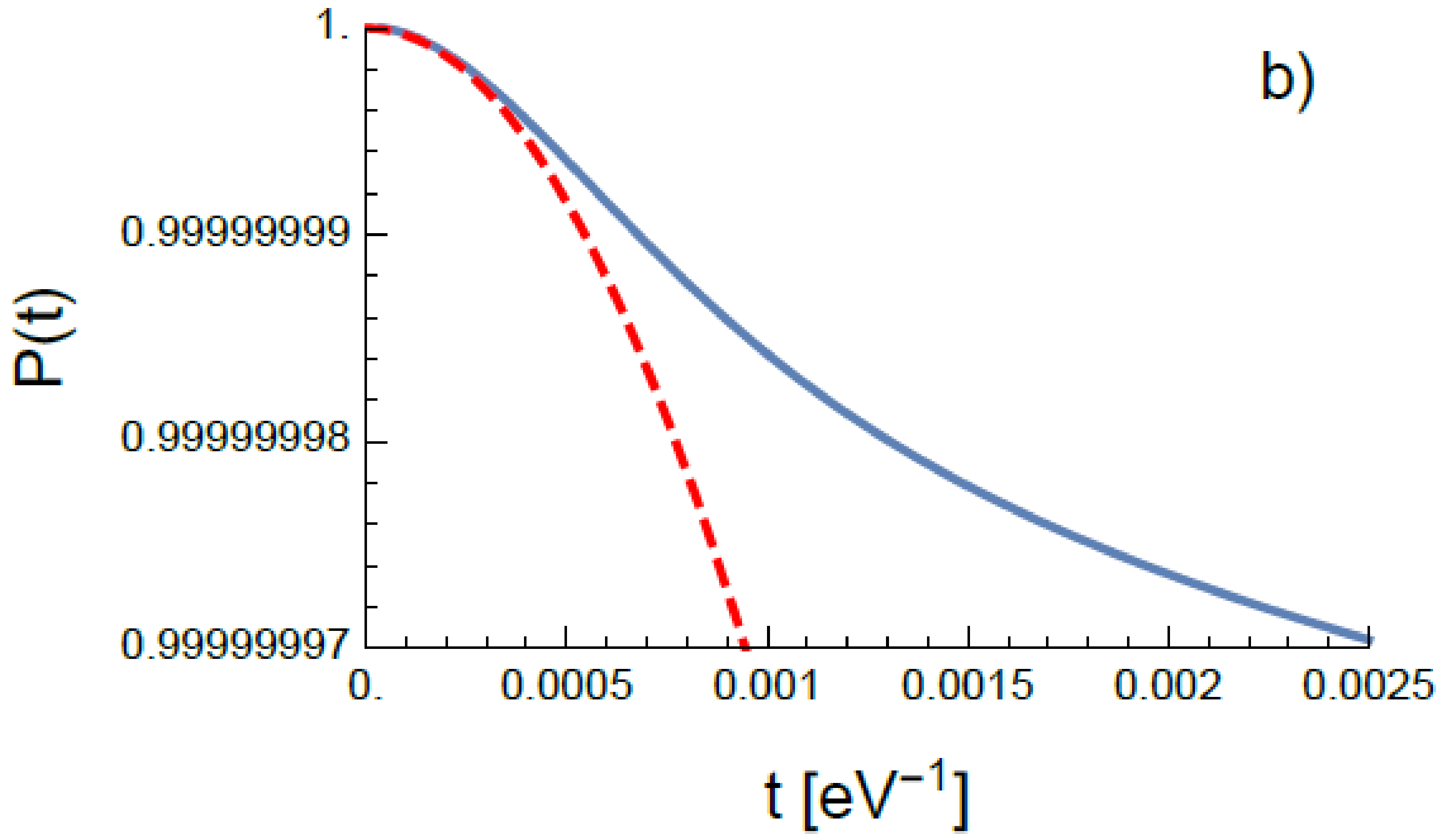
Survival probability at very short times/1

$$P(t) \simeq 1 - \frac{1}{2} \frac{\mathrm{d}^2 P(t)}{\mathrm{d}t^2}\bigg|_{t=0} t^2 + \ldots = 1 - \frac{t^2}{\tau_Z^2} + \ldots \; .$$

$$A(t) = 1 - it \langle E \rangle - \frac{t^2}{2} \langle E^2 \rangle + \ldots$$

$$\tau_Z = \sqrt{\frac{1}{\langle E^2 \rangle - \langle E \rangle^2}} = \frac{1}{\sigma_E} \simeq 5.45911 \, \mathrm{eV}^{-1} = 3.59325 \times 10^{-15} \, \mathrm{s} \; .$$

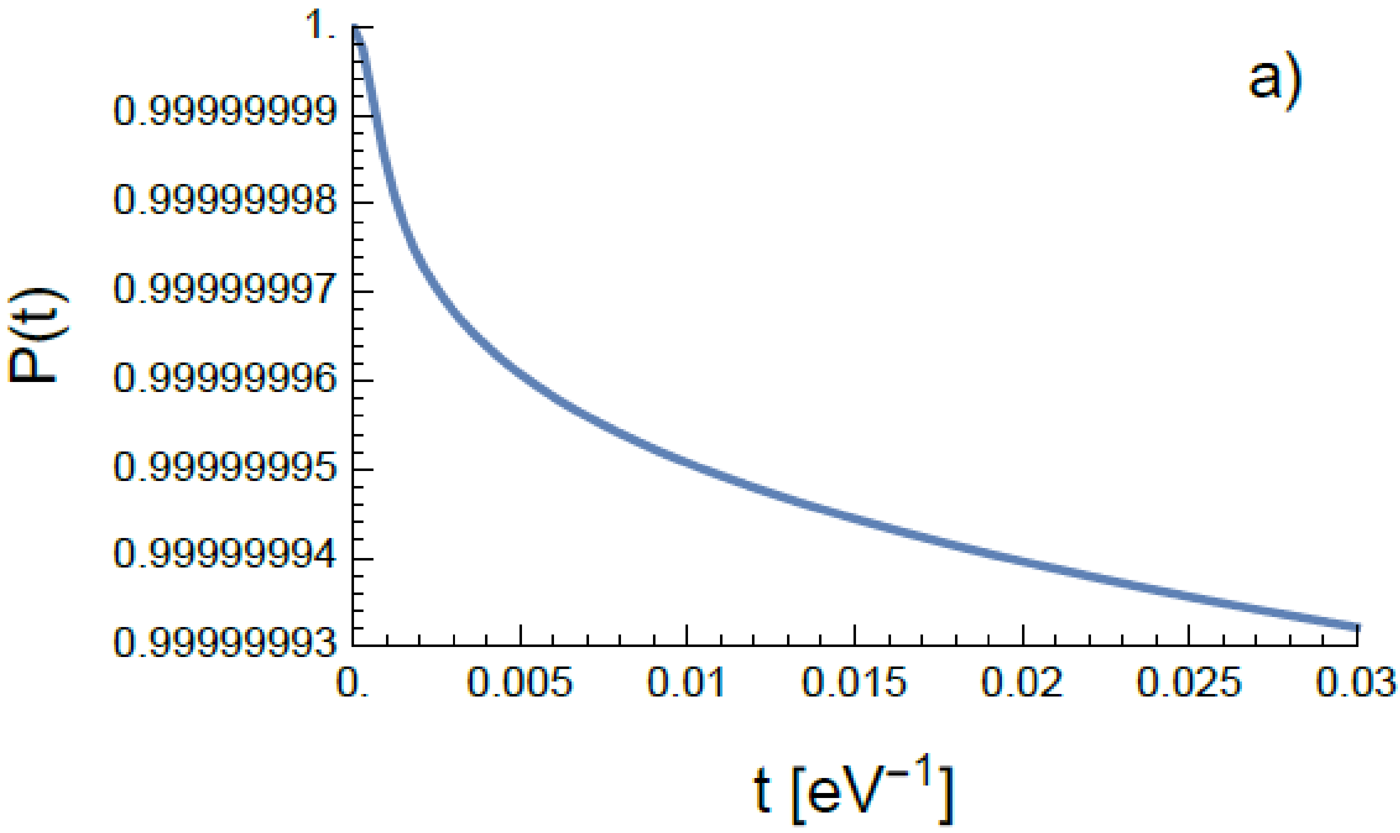
Survival probability at very short times/2



It is important to stress that in the present case, the Zeno time τ_Z is actually much longer than the non-exponential region in general and the quadratic region in particular.

2408.06905 [quant-ph]

Survival probability at short times: anti-Zeno



2408.06905 [quant-ph]

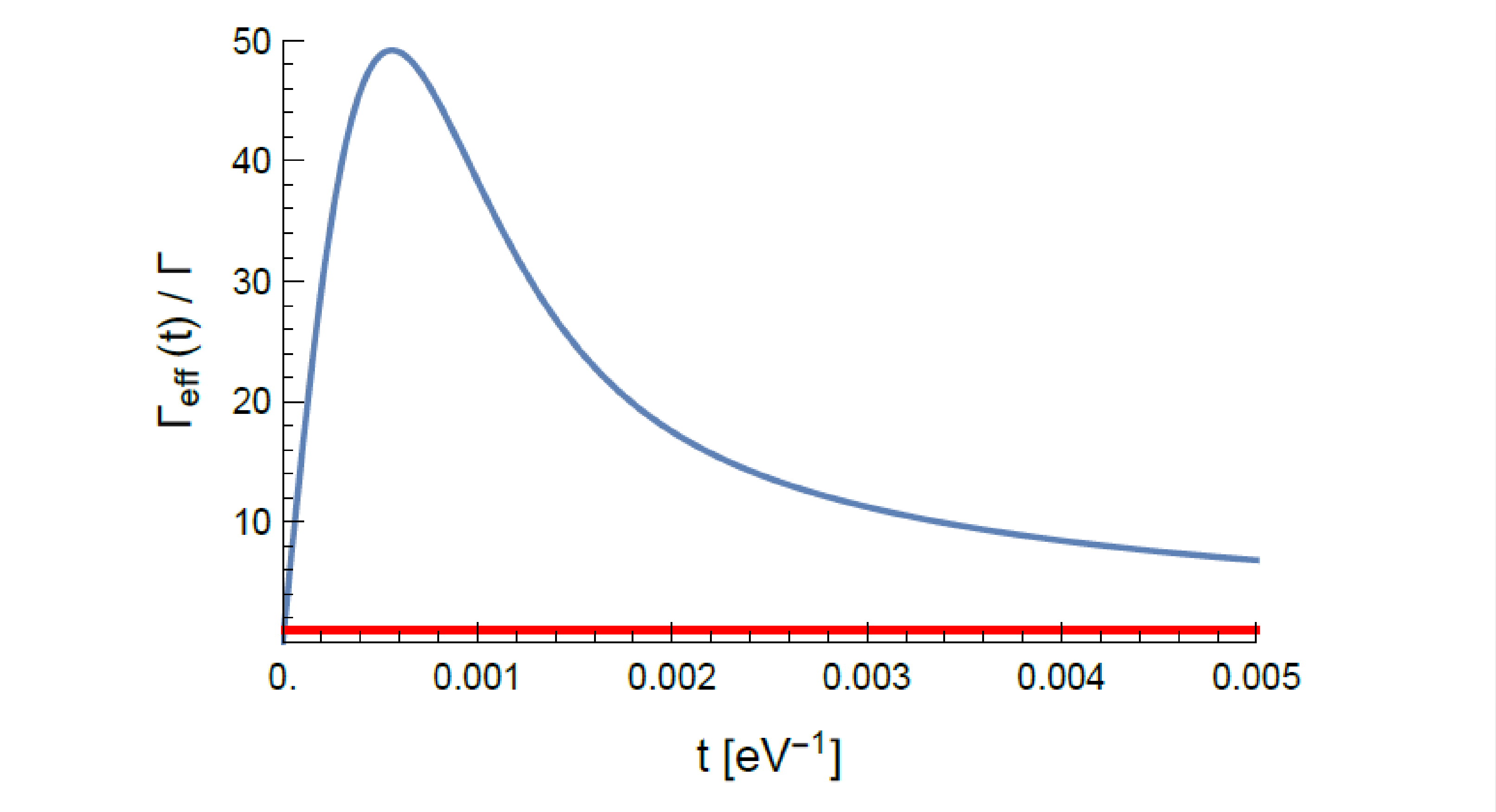
“Effective” decay width/1

$$\Gamma_{\text{eff}}(t) = -\frac{\mathrm{d}P(t)}{\mathrm{d}t}\frac{1}{P(t)} \, .$$

Table 1: Selected numerical values of the effective decay width within the anti-Zeno domain together with the corresponding times.

$\Gamma_{\text{eff}}(t)/\Gamma$	Time in eV^{-1}	Time in s
2	0.02130	1.40183×10^{-17}
1.1	0.06242	4.10857×10^{-17}
1.01	0.08234	5.41941×10^{-17}

“Effective” decay width/2



2408.06905 [quant-ph]

Survival amplitude at late times/2

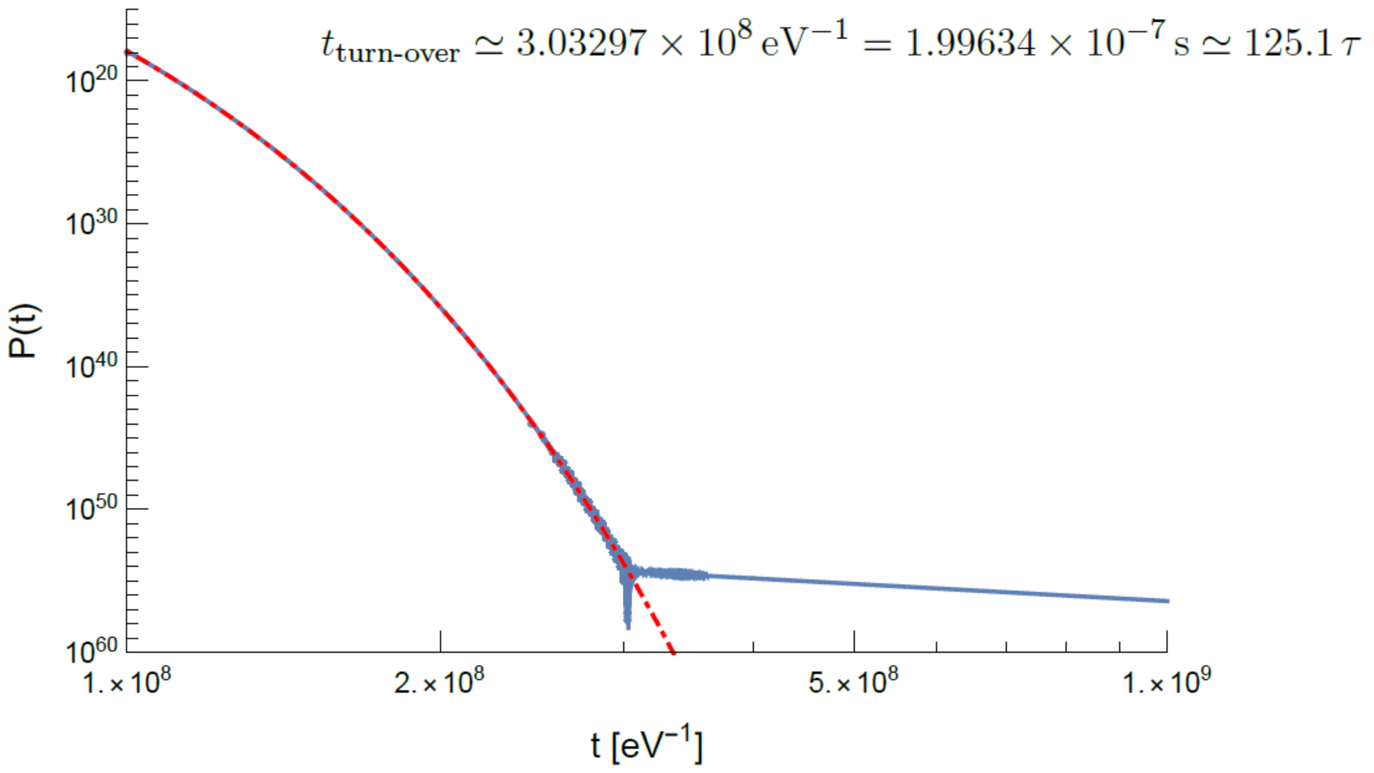


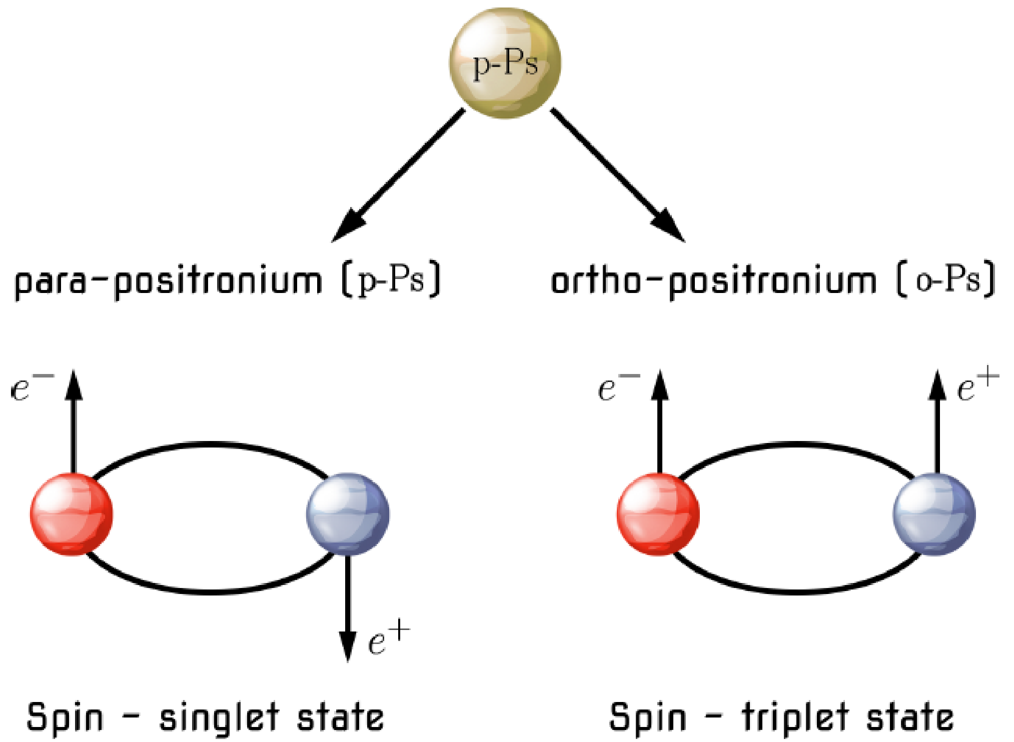
Figure 5: Survival probability at long times in log-log form. The red curve corresponds to purely exponential decay. An interesting feature is given by the fast oscillations close to the turn-over time.

2408.06905 [quant-ph]

Lesson: non-exp. decay is very hard to spot for ‘simple’ systems!

Part 2: positronium

Positronium (Ps): non-relativistic electron-positron bound state



PARA-POSITRONIUM (p-Ps)

Mass of positronium

$$2m_e - \left(\frac{\alpha^2 m_e}{4}\right)$$

m_e- mass of the electron $\qquad \alpha-$ fine structure constant

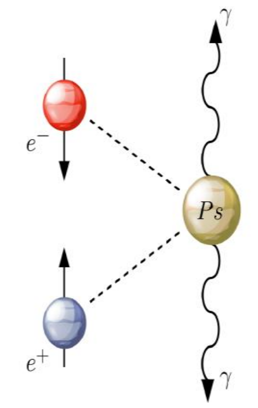
Quantum numbers

Non-relativistic notation	relativistic notation
$n\ {}^{2S+1}L_J = 1\ {}^1S_0$	$J^{PC} = 0^{-+}$

Wave function

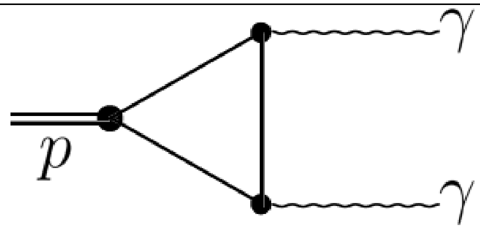
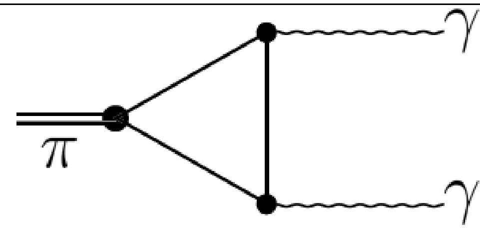
$$\psi(\vec{x}) = \frac{1}{(\pi a^3)^{1/2}} e^{-r/a}$$

$a-$ twice the Bohr radius of atomic hydrogen



W.f. in momentum space: $$A(\vec{q}) = \left(1 + \frac{4}{\alpha^4 + m_e^2}\vec{q}^2\right)^{-2}$$

Uniwersytet Jana Kochanowskiego w Kielcach

PARA-POSITRONIUM (p-Ps)	**PION (π^0)**
Ground state for electron-positron system $(n=1)$	Ground state for quark-antiquark system $(n=1)$
Non-relativistic state	Relativistic state (Goldstone boson)
$J^{PC}=0^{-+}$	$J^{PC}=0^{-+}$
Decays into $\gamma\gamma$	Decays into $\gamma\gamma$
 electrons are going around	 quarks are going around

e^- propagator $\neq$ quark propagator

Yet, in first approximation quark is taken

as a free propagator

$$\Gamma(\text{p-Ps} \to 2\gamma) = \frac{\alpha^5 m_e}{2}$$

Theory*	Experiment**
$8032.5028(1)\ \mu s^{-1}$	$7990.9(1.7)\ \mu s^{-1}$

mean lifetime of ~ 0.12 ns

* J.A. Wheeler, Ann. N.Y. Acad. Sci. 48, 219 (1946).
J. Pirenne, Arch. Sci. Phys. Nat. 29, 265 (1947)
** Al-Ramadhan, A. H., and D. Gidley (1994), Phys. Rev. Lett. 72, 1632.

At the lowest order it becomes:

$$\Gamma\left({}^1S_0 \to 2\gamma\right) = \frac{1}{2}\frac{e^4|\psi(\vec{x}=0)|^2}{\pi m^4}\int_0^\infty |\vec{k}_1|^2\delta(2m-2|\vec{k}_1|)d|\vec{k}_1| =$$

$$= \frac{e^4|\psi(\vec{x}=0)|^2}{4^2} = \frac{4\pi\alpha^2}{m^2}|\psi(\vec{x}=0)|^2$$

$|\psi(\vec{x}=0)|^2 \sim \alpha^3$
$|\psi(\vec{x}=0)| \sim \alpha^{3/2}$

QED, order by order

- One loop level

$$\Gamma(\text{p-Ps} \to \gamma\gamma) = \Gamma_0 \left\{ 1 + \frac{\alpha}{\pi} \left(\frac{\pi^2}{4} - 5 \right) \right\}$$

$$= 7985.249 \mu s^{-1}$$

I. Harris and L.M. Brown, Phys. Rev. 105, 1656 (1957)

- Two loop level

$$\Gamma_{\text{p-Ps}} = \Gamma_0 \left\{ -2\alpha^2 ln\alpha + B_{2\gamma} \left(\frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3}{2\pi} ln^2\alpha + C\frac{\alpha^3}{\pi} ln\alpha + D \left(\frac{\alpha}{\pi} \right)^3 \right\}$$

$$= 7989.6178(2) \mu s^{-1}$$

G. S. Adkins, N. M. McGovern, R. N. Fell and J. Sapirstein, Phys. Rev. A **68** (2003), 032512
A. Czarnecki and S. G. Karshenboim, [arXiv:hep-ph/9911410 [hep-ph]].

Results (partly) based on:

ACTA PHYSICA POLONICA A No. 5 Vol. **146** (2024)

Proceedings of 5th Jagiellonian Symposium on Advances in Particle Physics and Medicine, Kraków, Poland, 2024

Two-Photon Decay of Para-Positronium Within a Composite Approach

M. Piotrowska[a,*] and F. Giacosa[a,b]

[a]*Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406 Kielce, Poland*
[b]*Institute for Theoretical Physics, J.W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt, Germany*

Doi: 10.12693/APhysPolA.146.699 *e-mail: milena.piotrowska@ujk.edu.pl

Compositeness condition

PHYSICAL REVIEW VOLUME 137, NUMBER 3B 8 FEBRUARY 1965

Evidence That the Deuteron Is Not an Elementary Particle*

STEVEN WEINBERG†
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 30 September 1964)

If the deuteron were an elementary particle then the triplet n-p effective range would be approximately $-ZR/(I-Z)$, where $R=4.31$F is the usual deuteron radius and Z is the probability of finding the deuteron in a bare elementary-particle state. This formula is model-independent, but has an error of the order of the range $m_\pi^{-1}=1.41$F of the n-p force, so it becomes exact only in the limit of small deuteron binding energy, i.e., $R \gg m_\pi^{-1}$. The experimental value of the effective range is not of order R and negative, but rather of order m_π^{-1} and positive, so Z is small or zero and the deuteron is mostly or wholly composite.

THE LAGRANGIAN

$$\mathcal{L}_{int} = g_P P(x)\bar{\psi}(x)i\gamma^5\psi(x) - eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

- $P(x)$ is the pseudoscalar positronium field
- $\psi(x)$ is the electron field
- $A_\mu(x)$ is the photon field
- e is the electric charge of the proton
- g_P is the positronium-constituent coupling constant

The Lagrangian contains at the same time the bound state and its constituents.
It is not a fundamental Lagrangian.

COMPOSITE MODEL

Positronium (Ps) is a bound state.

How to describe it?

WEINBERG COMPOSITENESS CONDITIONS

(The positronium is not an elementary object, just as the deuteron)

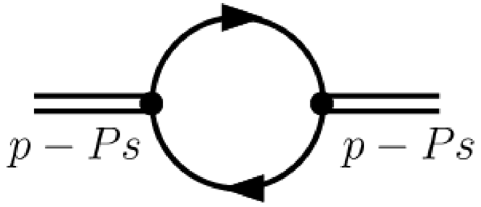
PARA-POSITRONIUM

- form factor $\sim$ wave function
- coupling constant (g) is fixed:

$$g_P = \sqrt{\frac{1}{\Sigma'(s=M_p^2)}}$$

$\Sigma(s = M_p^2)$- the loop function

- Loop diagram



THE LAGRANGIAN

$$\mathcal{L}_{int} = g_P P(x)\bar{\psi}(x) i\gamma^5 \psi(x) - eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

- $P(x)$ is the pseudoscalar positronium field
- $\psi(x)$ is the electron field
- $A_\mu(x)$ is the photon field
- e is the electric charge of the proton
- g_P is the positronium-constituent coupling constant

TRIANGLE AMPLITUDE I

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{\mathcal{F}(q,p)}{\left(q_1^2 - m_e^2 + i\varepsilon\right)\left(q_2^2 - m_e^2 + i\varepsilon\right)\left(q_3^2 - m_e^2 + i\varepsilon\right)}$$

Solved by using two independent methods:

- WICK ROTATION METHOD
- RESIDUE THEOREM

This is not a fundamental theory!

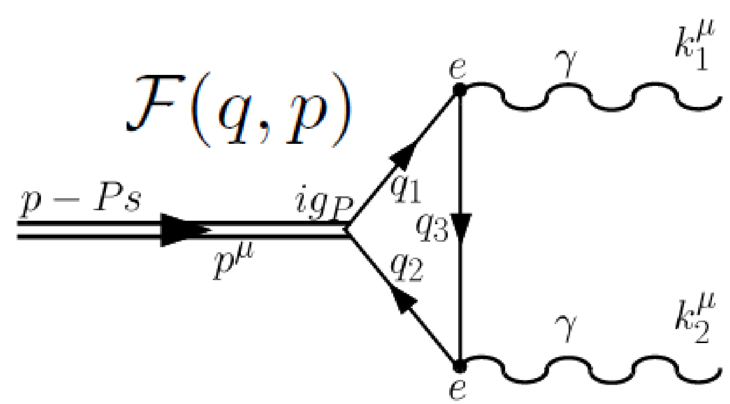
Vertex function is important: $\mathcal{F}(q,p)$

What about the coupling constant?

TRIANGLE AMPLITUDE

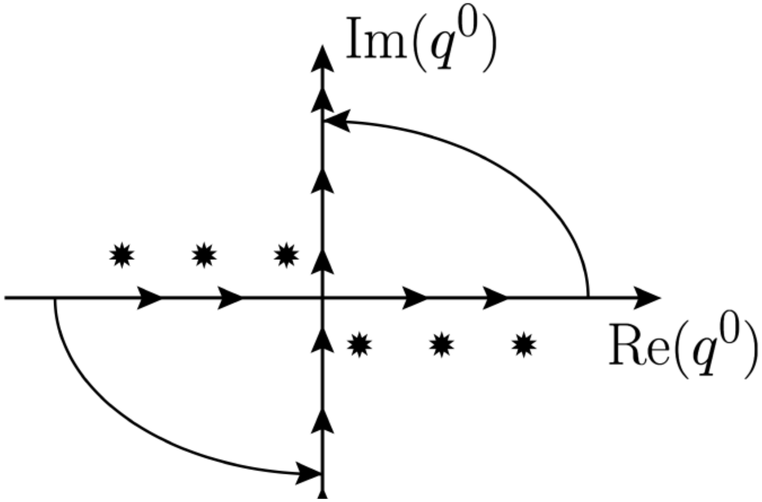
$$I = i \int \frac{d^4q}{(2\pi)^4} \frac{\mathcal{F}(q,p)}{D_1 D_2 D_3}$$

TRIANGLE DIAGRAM



Poles

$$I = \int \frac{d^3 q}{(2\pi)^3} \left[\int \frac{dq^0}{2\pi} \frac{\mathcal{F}}{D_1 D_2 D_3} \right]$$



By setting $D_{1,2,3}$=0 one gets:

Poles of D_1	Poles of D_2
$L_1 = -\frac{M_P}{2} - \sqrt{\rho^2 + q_z^2 + m_e^2} + i\delta$ $R_1 = -\frac{M_P}{2} + \sqrt{\rho^2 + q_z^2 + m_e^2} - i\delta$	$L_2 = \frac{M_P}{2} - \sqrt{\rho^2 + q_z^2 + m_e^2} + i\delta$ $R_2 = \frac{M_P}{2} + \sqrt{\rho^2 + q_z^2 + m_e^2} - i\delta$
Poles of D_3	
$L_3 = -\sqrt{\rho^2 + (q_z - k_z)^2 + m_e^2} + i\delta$ $R_3 = \sqrt{\rho^2 + (q_z - k_z)^2 + m_e^2} - i\delta$	

What is the vertex function?

- NAIVE ANSATZ 1: $\mathcal{F}(q,p)=\mathcal{F}(\vec{q}^2)=A(\vec{q})$

$$\mathcal{F}(q,p)=\mathcal{F}(\vec{q}^2)=A(\vec{q})=\left(1+\frac{4}{\alpha^4+m_e^2}\vec{q}^2\right)^{-2}$$

RESULTS:

-Analytical formula for the non-relativistic limit is correct (necessary condition):

$$\Gamma_{p\to\gamma\gamma}=\frac{\alpha^5 m_e}{2}$$

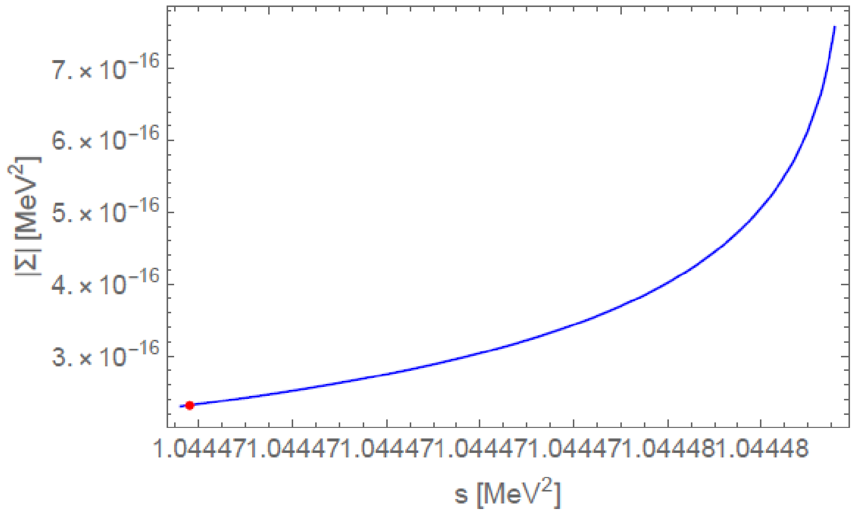
$-\Gamma_{p\to\gamma\gamma}$ (by evaluating the full integrals) turns out to be too small (factor 2) when compared to the experimental value.

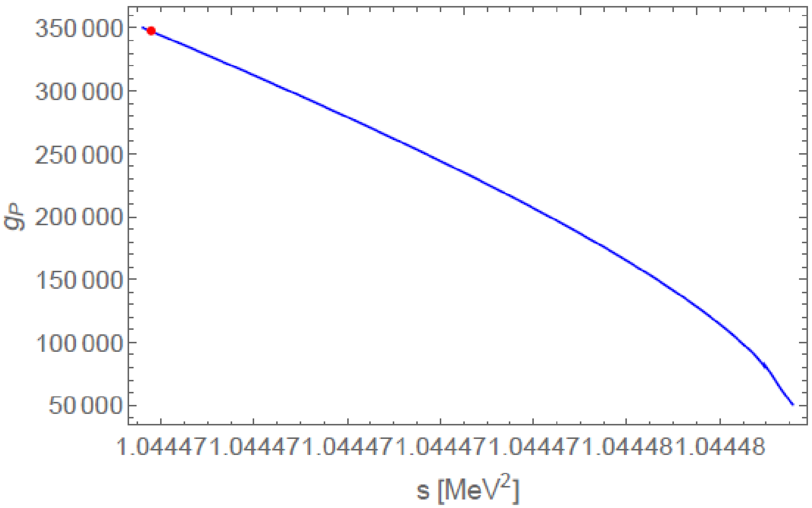
CONCLUSIONS:

-Assumption of momentum wave function as form factor is not correct.
-Yet, vertex function must behave as the wave function for small momenta.

◀ PESTIEAU ANSATZ 2*: $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2}\left(\vec{q}^2+\gamma^2\right)$

with $\gamma^2 = m^2 - \frac{M_P^2}{4}$





* J. Pestieau, C. Smith and S. Trine, "Positronium decay: Gauge invariance and analyticity," Int. J. Mod. Phys. A **17** (2002), 1355-1398 doi:10.1142/S0217751X02009606

◀ PESTIEAU ANSATZ 2*: $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2}\left(\vec{q}^2+\gamma^2\right)$

with $\gamma^2 = m^2 - \frac{M_P^2}{4}$

RESULTS

PARA-POSITRONIUM	$\Gamma_{P-ps\to\gamma\gamma}\ [\mu s^{-1}]$
Experimental result**	$7990.9(1.7)$
pole 1	7968.2
pole 1 + pole 2	7995.3
pole 1 + pole 2 + pole 3	7920.3
Result of *	7952.7

* J. Pestieau, C. Smith and S. Trine, "Positronium decay: Gauge invariance and analyticity," Int. J. Mod. Phys. A **17** (2002), 1355-1398 doi:10.1142/S0217751X02009606
** Al-Ramadhan, A. H., and D. Gidley (1994), Phys. Rev. Lett. 72, 1632.

Our Ansatz: additional relativistic correction

- OUR ANSATZ 3*: $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2}\left(\vec{q}^2+\gamma^2\right)\sqrt{\vec{q}^2+m^2}$

  with $\gamma^2 = m^2 - \frac{M_P^2}{4}$

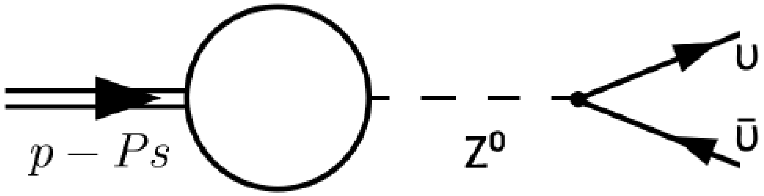
Motivation

Weak decay constant

$$f_p \sim \int d^3q A(\vec{q})$$

$$f_p \sim \int d^3q \frac{\mathcal{F}(\vec{q})}{\sqrt{\vec{q}^2+m^2}(\vec{q}^2+\gamma^2)}$$

$$\mathcal{F}(\vec{q}) = \sqrt{\vec{q}^2+m^2}\left(\vec{q}^2+\gamma^2\right)A(\vec{q})$$



RESULTS

PARA-POSITRONIUM	$\Gamma_{P-ps\to\gamma\gamma}\ [\mu s^{-1}]$
Experimental result**	$7990.9(1.7)$
pole 1	8057.65
pole 1 + pole 2	8121.07
pole 1 + pole 2 + pole 3	7981.45

Discussion

- Why does it seem to work?
- The approach effectively resums up to infinite order a certain class of QED diagram.
- The hard part is thrown into the p-ps-(anti)electron vertex
- It is ‘quite’ easy to implement.

Application to charm-anticharm $\eta c(1S)$

Same approach. Just rescale. Mass mc = 1.65 GeV.

Same w.f.

- $\Gamma(\eta c(1S) \rightarrow \gamma\gamma) = 3.9 \cdot 10^{-6}$ GeV
- $\Gamma(\eta c(1S) \rightarrow \gamma\gamma$, exp.$) = (5.1 \pm 0.4) \cdot 10^{-6}$ GeV
- Why theory a bit smaller? It is clear: in reality the charmonium is a bit more squeezed in space, thus broader in momentum.

Conclusions

- 2p-1S transition of H-atom as example of non-exp decay: it shows how hidden are such interesting phenomena!

- Para-positronium into two photons: composite approach. It is quite easy and it works.

- P-ps(2S) (Decay width 0.12·p-ps(1S))

- η_c meson, η_b meson,…(both 1S and 2S)

- Decay law of p-ps? O-ps ?

Thanks!

BW with threshold **properly done**

We assume that:

$$\operatorname{Im}\Pi(E) = \begin{cases} \frac{\Gamma}{2} \text{ for } E \in (E_{th}, \Lambda) \\ 0 \text{ otherwise} \end{cases}$$

<E>=lnΛ=∞

In the limit $\Lambda \to \infty$ and by using one subtraction we get:

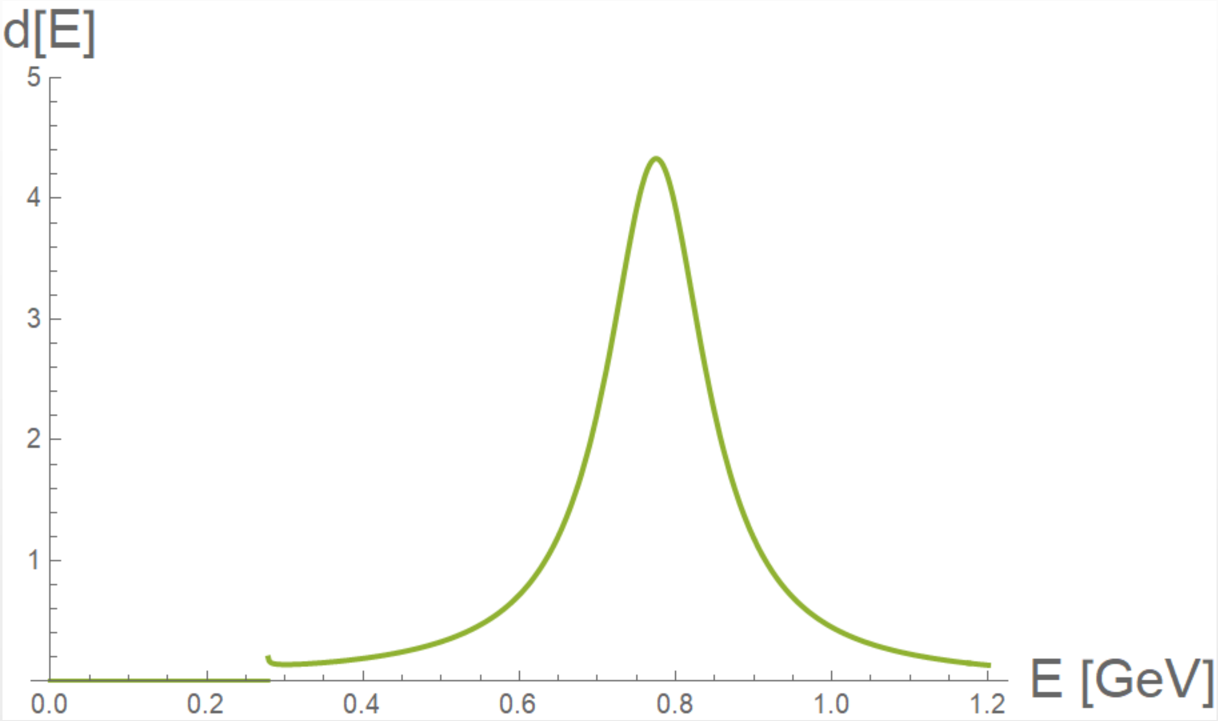
$$\Pi(E) = \frac{\Gamma}{2\pi} \ln\left(\frac{-E_{th} + M}{E_{th} - E}\right)$$

Then:

$$d_S(E) = \frac{\Gamma}{2\pi} \frac{1}{\left[E - M + \frac{\Gamma}{2\pi} \ln\left(\frac{M - E_{th}}{E_{th} - E}\right)\right]^2 + \frac{\Gamma^2}{4}} \theta(E - E_{th})$$

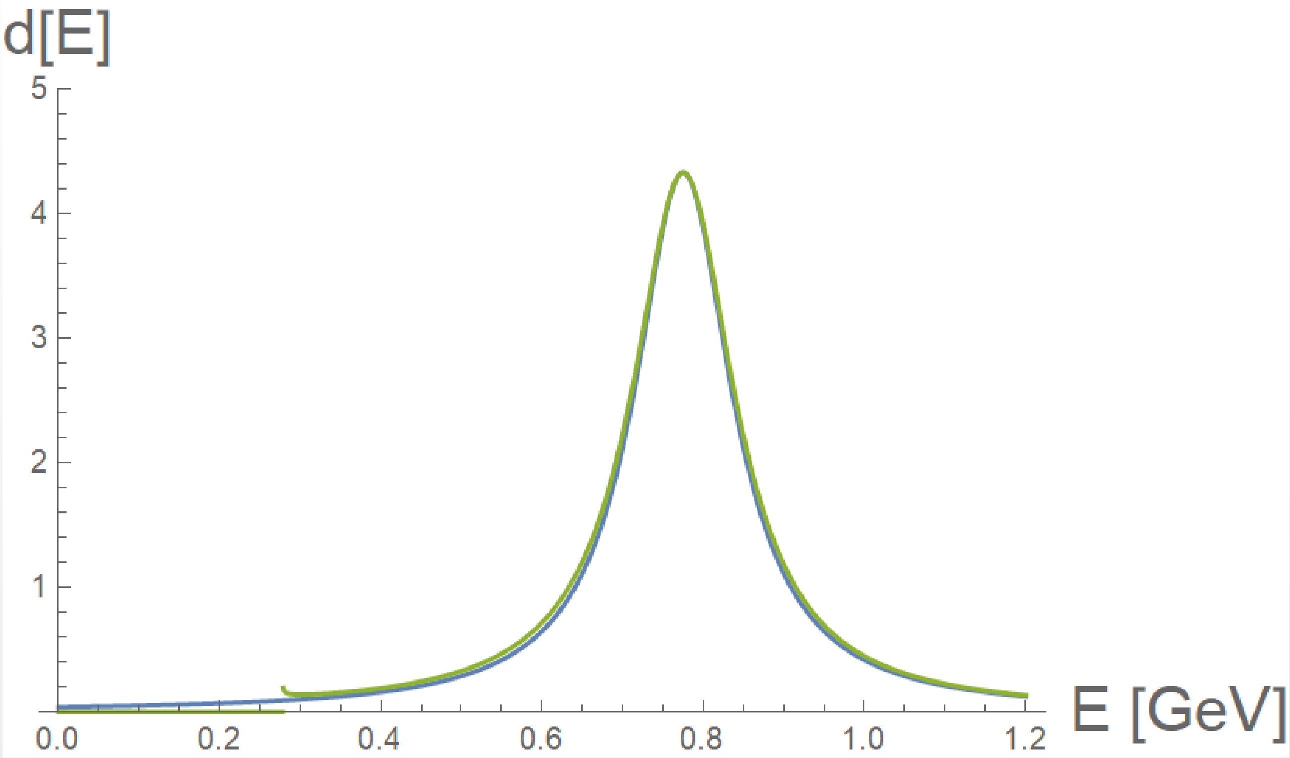
This is actually the correct Breit-Wigner with threshold!
Correctly normalized to unity, no need of an extra N…but somewhat not handy

BW with threshold **properly done**



Correct BW with threshold

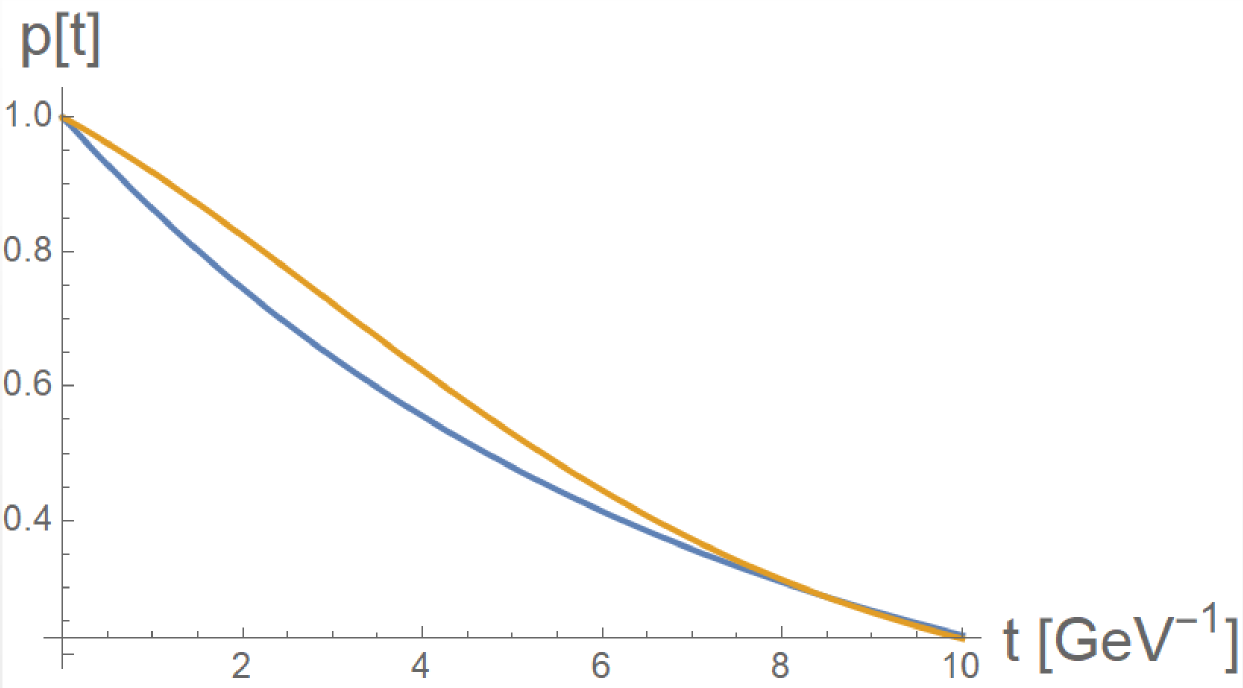
BW with threshold **properly done**



Comparison with plain BW: indeed very similar around the peak!

Yet: unphysical behavior at threshold!

BW with threshold **(properly done)** and time-evolution



Blue: BW, yellow: BW with threshold (properly done)

$\langle E \rangle = \ln\Lambda = \infty$

Sill distribution

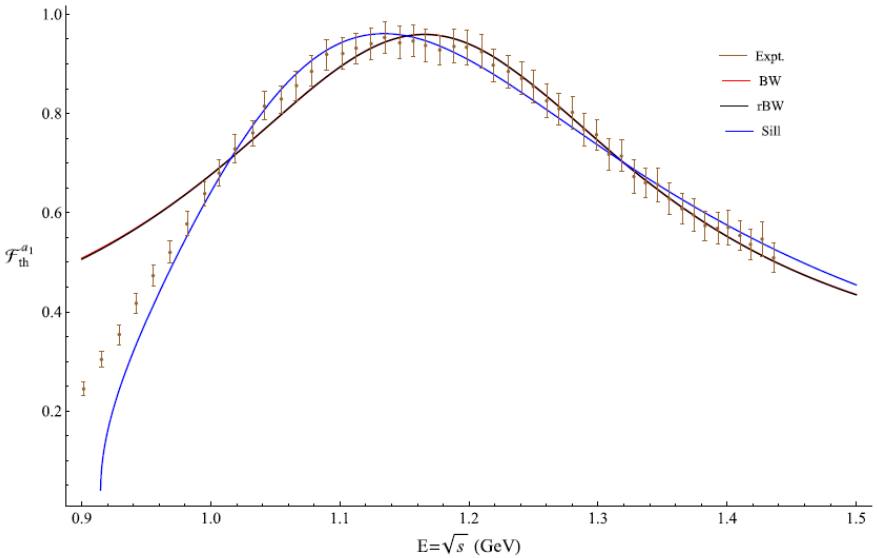
Eur. Phys. J. A (2021) 57:336
https://doi.org/10.1140/epja/s10050-021-00641-2
THE EUROPEAN PHYSICAL JOURNAL A
Regular Article - Theoretical Physics

A simple alternative to the relativistic Breit–Wigner distribution

Francesco Giacosa[1,2], Anna Okopińska[1], Vanamali Shastry[1,a]

Uniwersytet Jana Kochanowskiego w Kielcach

Example: $a_1(1230)$ meson



Distribution	M (MeV)	Γ (MeV)	χ^2/d.o.f	$\sqrt{s_{pole}}$(MeV)
Nonrelativistic BW	1165.6 ± 1.2	415 ± 15	4.31	$1166 - i\,208$
Relativistic BW	1146.5 ± 1.6	424 ± 16	4.25	$1165 - i\,209$
Sill	1181.3 ± 3.4	539 ± 27	3.52	$1046 - i\,250$

Applications in e.g.

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

ALICE

CERN-EP-2023-106
30 May 2023

Accessing the strong interaction between Λ baryons and charged kaons with the femtoscopy technique at the LHC

ALICE Collaboration*

$\Xi(1620)$ $\quad I(J^P) = \frac{1}{2}(?^?)$ Status: ∗
J, P need confirmation.

OMITTED FROM SUMMARY TABLE
What little evidence there is consists of weak signals in the $\Xi\pi$ channel. A number of other experiments (e.g., BORENSTEIN 72 and HASSALL 81) have looked for but not seen any effect.

$\Xi(1620)$ MASS

VALUE (MeV)	EVTS	DOCUMENT ID	TECN	COMMENT
≈ 1620 OUR ESTIMATE				

Francesco Giacosa

[arXiv:2106.03749 [hep-ph]].

https://doi.org/10.1140/epja/s10050-021-00641-2
Physical Journal A
Regular Article - Theoretical Physics

A simple alternative to the relativistic Breit–Wigner distribution

Francesco Giacosa[1,2], Anna Okopińska[1], Vanamali Shastry[1,a]

Uniwersytet Jana Kochanowskiego w Kielcach

See F.G., V. Shastry and A. Okopinska[arXiv:2106.03749 [hep-ph]].
and also [arXiv:2310.06346 [hep-ph]] for the non-rel limit

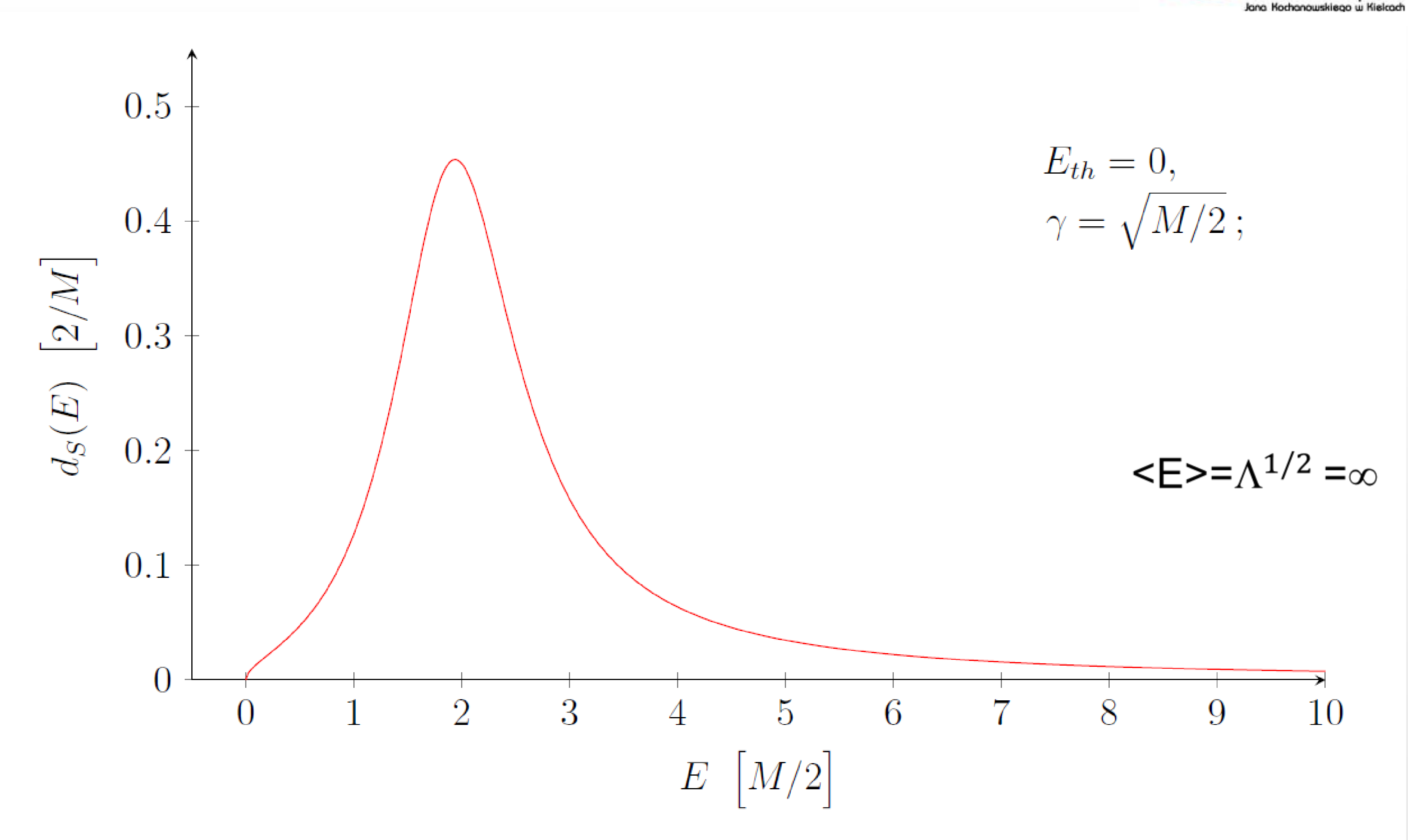
$$\mathrm{Im}[\Pi(E)] = \frac{\gamma}{2}\sqrt{E - E_{th}}\,\theta(E - E_{th})$$

$$\Pi(E) = \frac{i\gamma\sqrt{E - E_{th}}}{2}$$

$$G_S(E) = \frac{1}{E - M + \frac{i\gamma\sqrt{E-E_{th}}}{2} + i\epsilon}$$

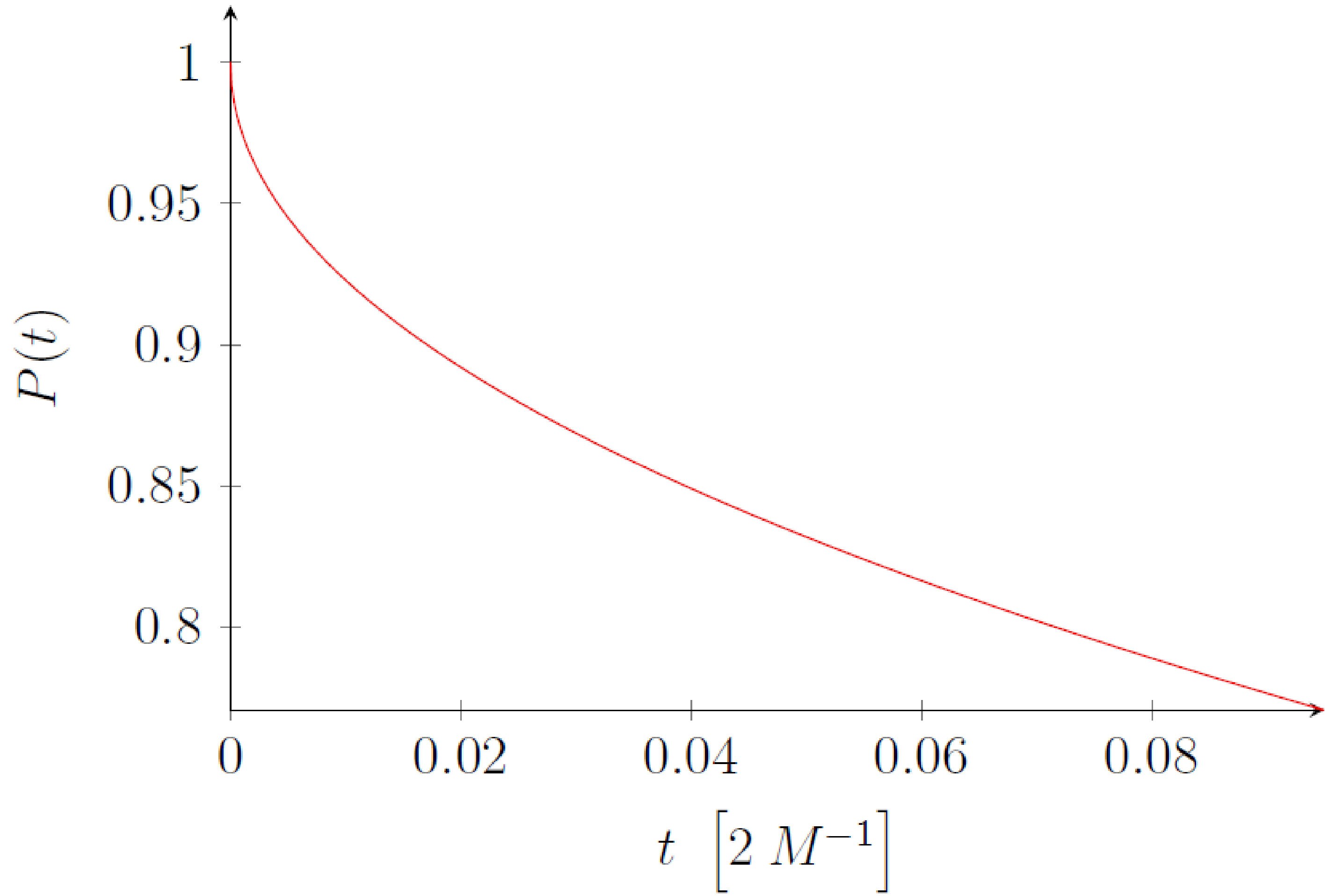
$$d_S(E) = \frac{\gamma\sqrt{E - E_{th}}}{2\pi}\frac{1}{(E - M)^2 + \frac{1}{4}(\gamma\sqrt{E - E_{th}})^2}$$

Non-rel. Sill distribution: ‘nice’ threshold behavior



K. Kyzioł, bachelor thesis, UJK Kielce, 2024

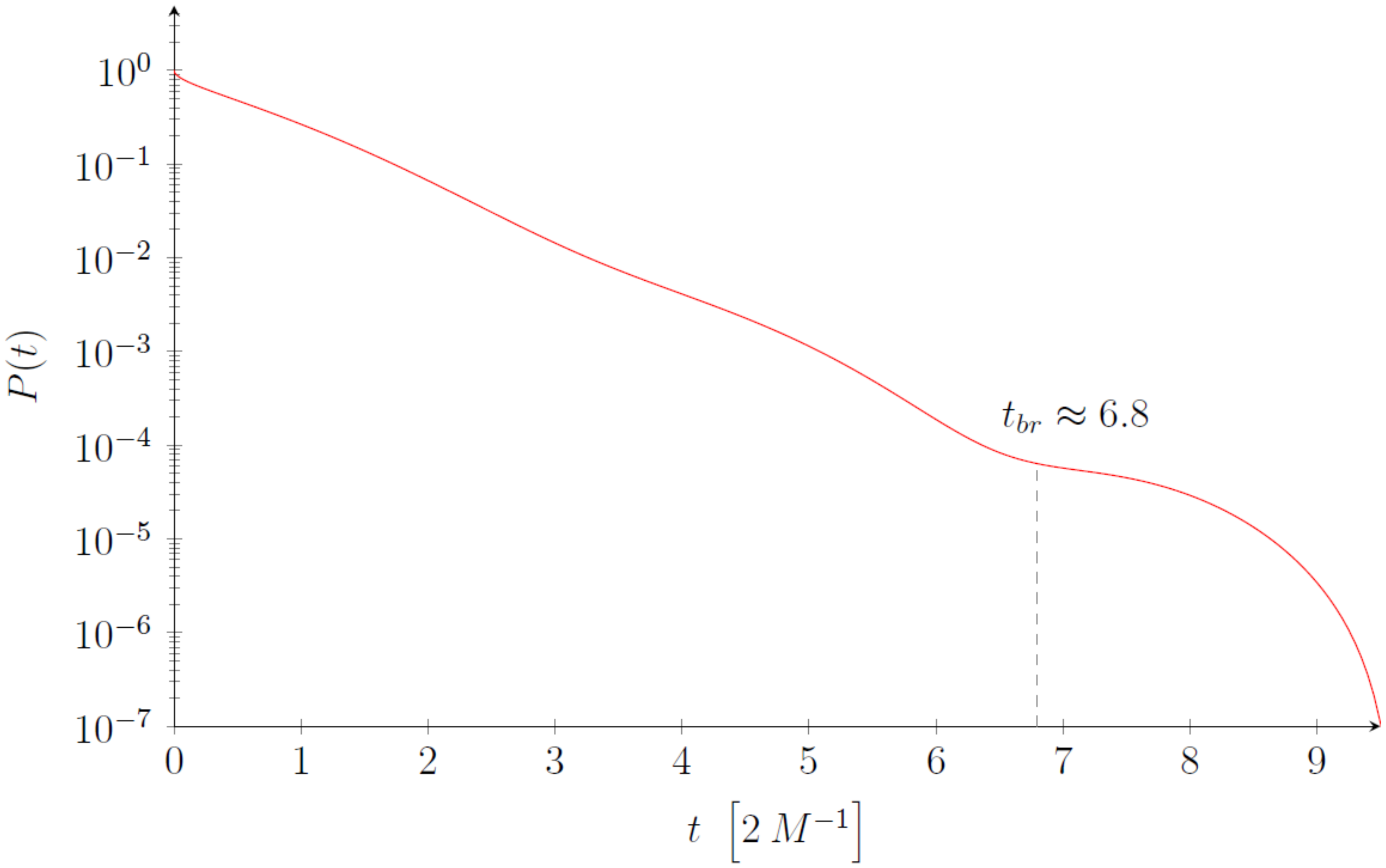
Plot of the survival probability nonrel Sill



K. Kyzioł, bachelor thesis, UJK Kielce, 2024

$<E>=\Lambda^{1/2}=\infty$

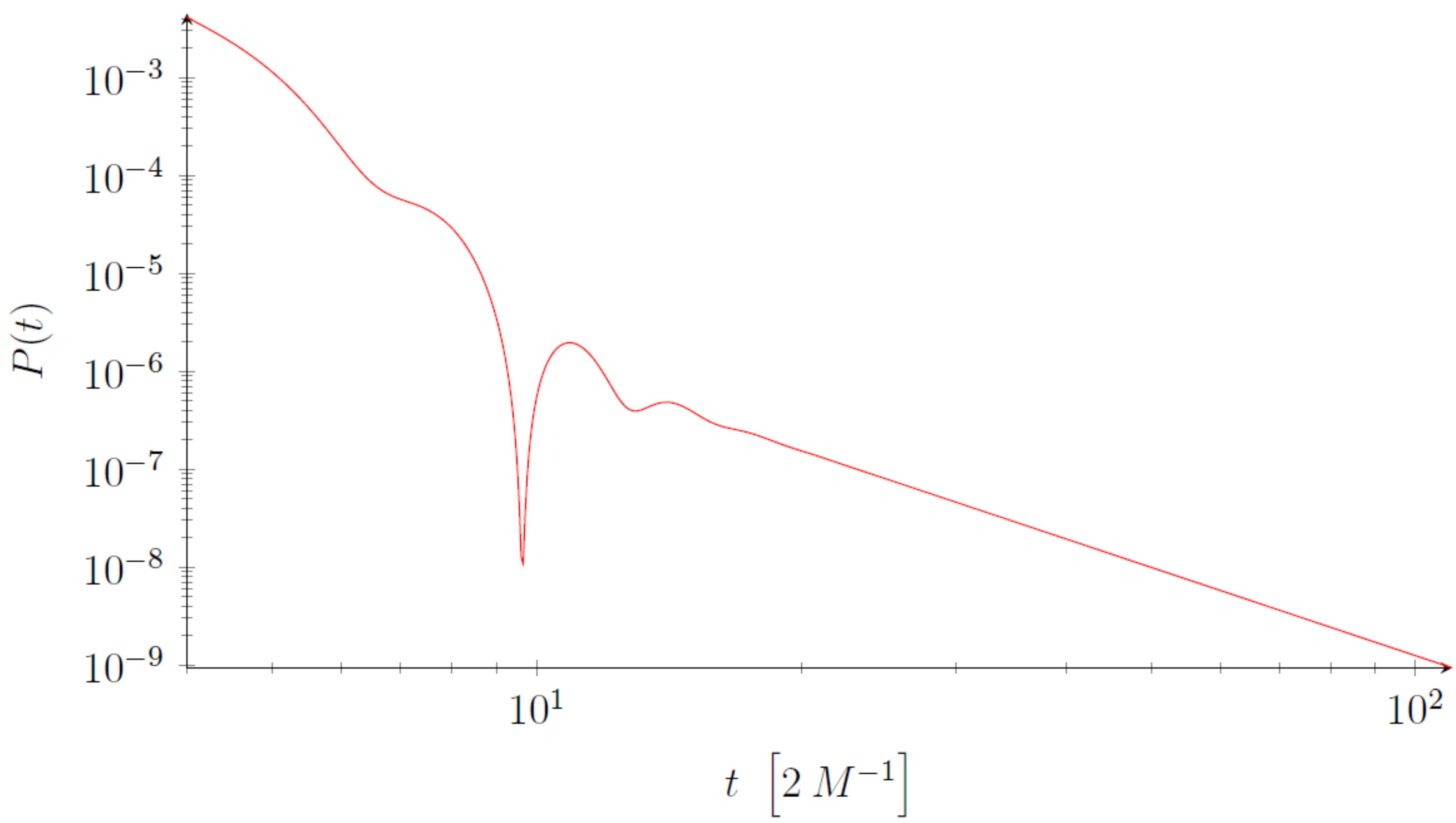
Log-plot



K. Kyzioł, bachelor thesis, UJK Kielce, 2024

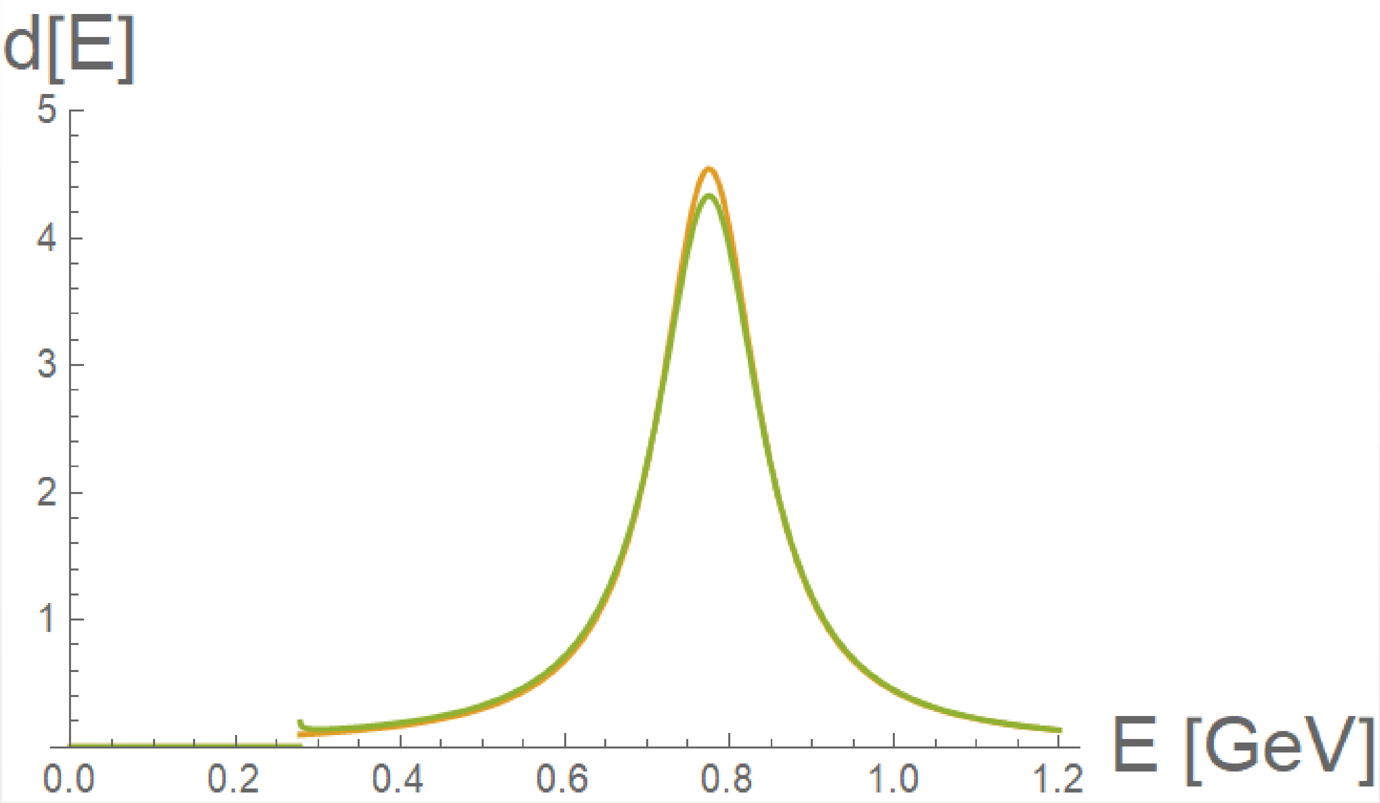
Log-Log-Plot

$$\ln P(t) \approx \ln(C \cdot t^{-3}) = \ln C - 3 \ln t \quad .$$



K. Kyzioł, bachelor thesis, UJK Kielce, 2024

BW with threshold **properly done**



Comparision with ‘naive’ BW with threshold

Relativistic Sill

Let us consider a resonance with mass M decaying into twoparticles:

$$E_{th} = m_1 + m_2 = \sqrt{s_{th}} \qquad s_{th} = E_{th}^2$$

We **assume** that:

$$\operatorname{Im}\Pi(s) = \sqrt{s - s_{th}}\,\tilde{\Gamma}\,\theta(s - s_{th})$$

$$\Gamma M = \tilde{\Gamma}\sqrt{M^2 - E_{th}^2}$$

Decay width as function of the energy:

$$\Gamma(s) = \frac{\sqrt{s - s_{th}}}{\sqrt{s}}\tilde{\Gamma}$$

Note, it saturates for large s

Relativsitic Sill

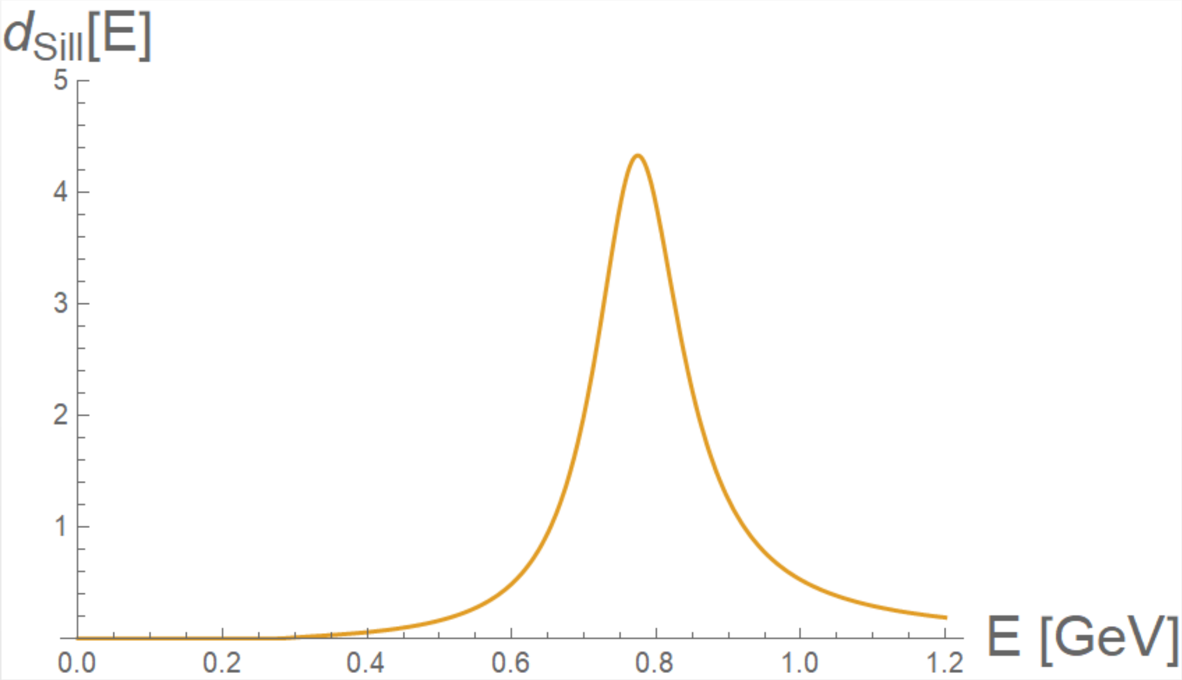
$$\Pi(s) = i\tilde{\Gamma}\sqrt{s - s_{th}}$$

$$G_S(s) = \frac{1}{s - M^2 + i\tilde{\Gamma}\sqrt{s - s_{th}} + i\varepsilon}$$

$$\begin{aligned} d_S(s) &= -\frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{s - M^2 + i\tilde{\Gamma}\sqrt{s - s_{th}} + i\varepsilon} \\ &= \frac{1}{\pi}\frac{\sqrt{s - s_{th}}\,\tilde{\Gamma}}{(s - M^2)^2 + (\sqrt{s - s_{th}}\,\tilde{\Gamma})^2}\theta(s - s_{th}) \end{aligned}$$

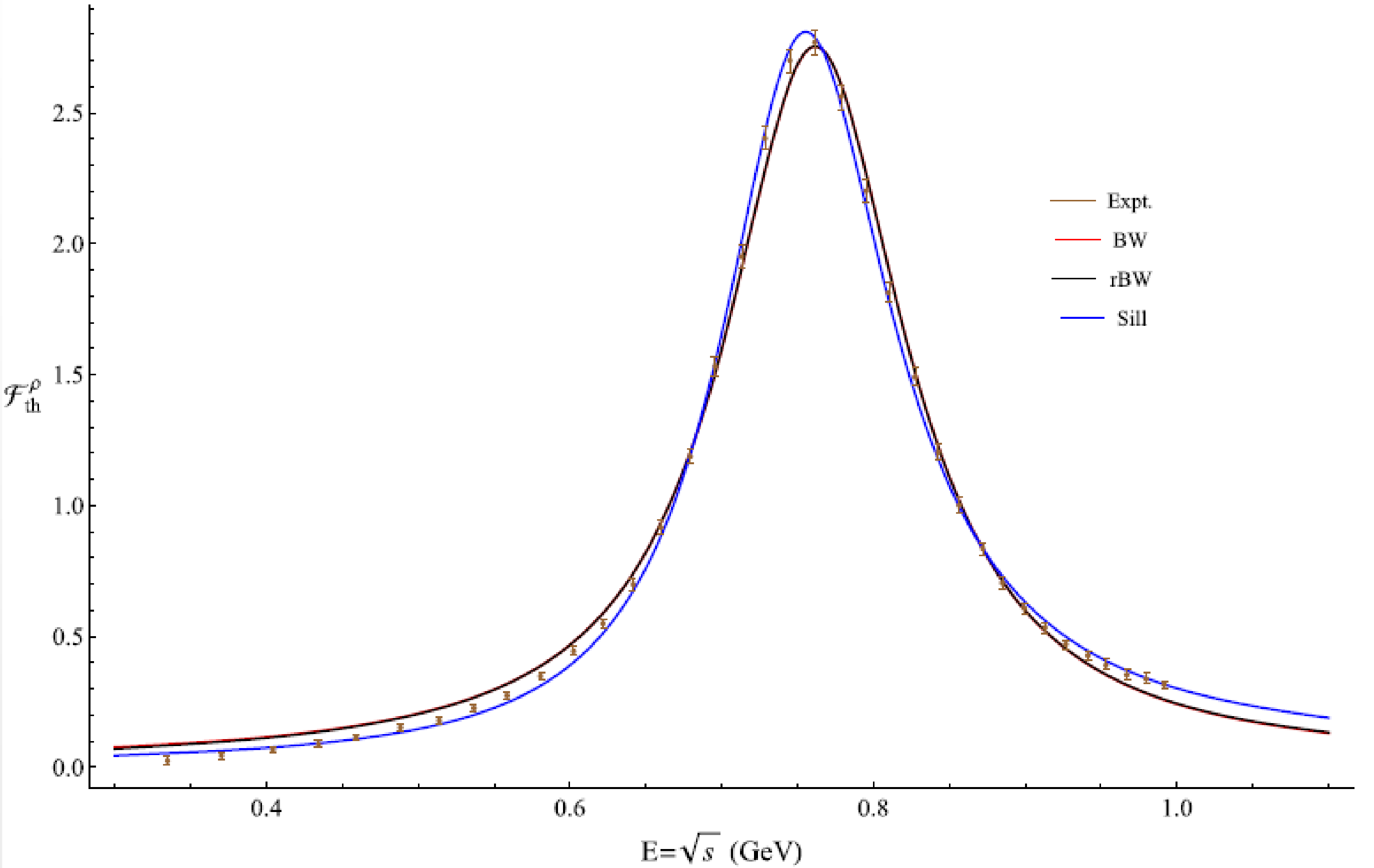
Relativistic Sill

$$d_S(E) = d_S^{\mathrm{Sill}}(E) \;=\; \frac{2E}{\pi} \frac{\sqrt{E^2 - E_{th}^2}\,\tilde{\Gamma}}{(E^2 - M^2)^2 + \left(\sqrt{E^2 - E_{th}^2}\,\tilde{\Gamma}\right)^2} \theta(E - E_{th})$$



Sill for the rho-meson

Rho meson



Aleph data for tau decay

Distribution	M (MeV)	Γ (MeV)	χ^2/d.o.f	$\sqrt{s_{pole}}$(MeV)
Nonrelativistic BW	761.64 ± 0.32	144.6 ± 1.3	10.16	$761.6 - i\,72.3$
Relativistic BW	758.1 ± 0.33	145.2 ± 1.3	9.42	$761.5 - i\,72.3$
Sill	755.82 ± 0.33	137.3 ± 1.1	3.52	$751.7 - i\,68.6$

Deviations from the exp. law at short times

Taylor expansion of the amplitude:

$$a(t)=\langle S|e^{-iHt}|S\rangle = 1-it\langle S|H|S\rangle-\frac{t^2}{2}\langle S|H^2|S\rangle+\ldots$$

$$a^*(t)=\langle S|e^{-iHt}|S\rangle = 1+it\langle S|H|S\rangle-\frac{t^2}{2}\langle S|H^2|S\rangle+\ldots$$

p(t) decreases quadratically (not linearly);

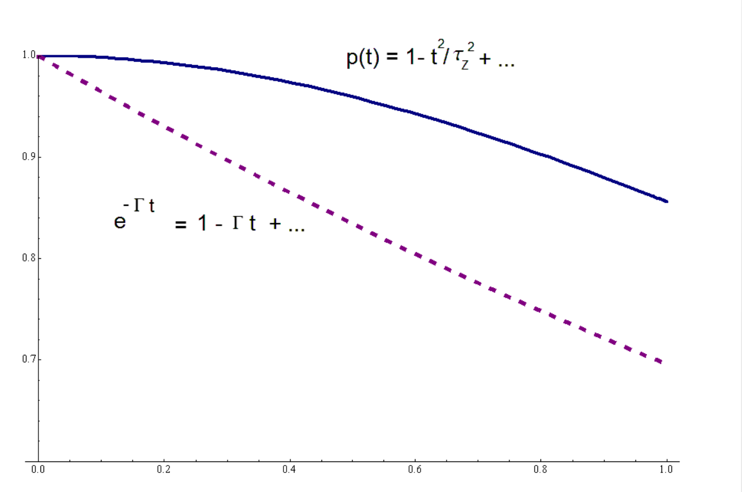
no exp. decay for short times.

τ_Z is the \`Zeno time´.

It follows:

$$\boxed{p(t)=|a(t)|^2=a^*(t)a(t)=1-t^2\left(\langle S|H^2|S\rangle-\langle S|H|S\rangle^2\right)+\ldots=1-\frac{t^2}{\tau_Z^2}+\ldots}$$

where $\tau_Z=\dfrac{1}{\sqrt{\langle S|H^2|S\rangle-\langle S|H|S\rangle^2}}$.



Note: the quadratic behavior holds
for any quantum transition, not only for decays.

BW: properties

BW-distribution: $$d_S^{\mathrm{BW}}(E) = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \frac{\Gamma^2}{4}}$$

Pole:

$$z_{pole}^{\mathrm{BW}} = M - i\Gamma/2$$

BW-propagator: $$G_S^{\mathrm{BW}}(E) = \frac{1}{E - M + i\Gamma/2 + i\varepsilon}$$

Link prop-dist: $$d_S^{\mathrm{BW}}(E) = -\frac{1}{\pi} \mathrm{Im}[G_S^{\mathrm{BW}}(E)] = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \frac{\Gamma^2}{4}}$$

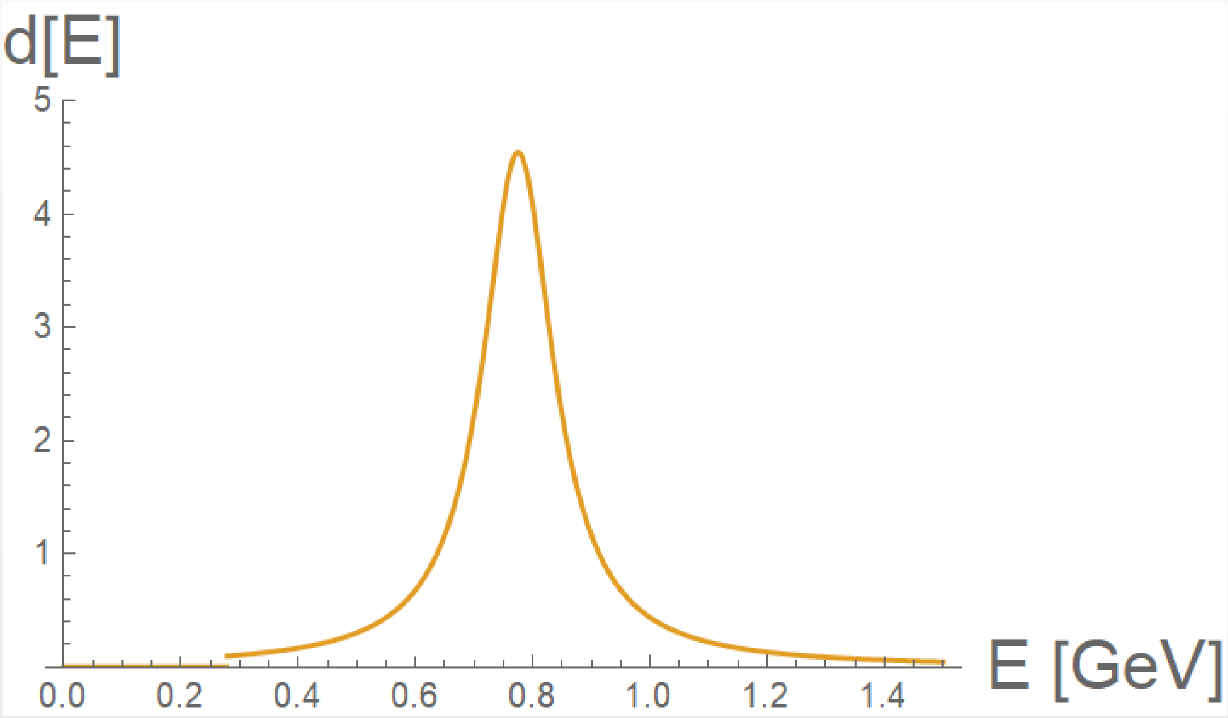
Normalization:
(important for prob. interpretation)

$$\int_{-\infty}^{+\infty} d_S^{\mathrm{BW}}(E) dE = 1$$

How to introudce a threshold?
BW with threshold (naive approach)

$$d_S(E) = N\frac{\Gamma}{2\pi}\frac{1}{(E-M)^2+\frac{\Gamma^2}{4}}\theta(E-E_{th})$$

N is needed because the normalization is lost!



Multichannel decay law

Physics Letters B 831 (2022) 137200

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

ELSEVIER

Multichannel decay law

Francesco Giacosa [a,b,*]

$w_1(t)$ is the probability that the decay has occurred in the first channel between (0,t)

$$\sum_{i=1}^{N} w_i = 1 - p(t)$$

ACTA PHYSICA POLONICA A No. 3 Vol. **142** (2022)

Proceedings of the 4th Jagiellonian Symposium on Advances in Particle Physics and Medicine

Multichannel Decay: Alternative Derivation of the i-th Channel Decay Probability

F. Giacosa [a,b,*]

[a] *Institute of Physics, Jan Kochanowski University, Uniwersytecka 7, 25-406, Kielce, Poland*

Uniwersytet Jana Kochanowskiego w Kielcach

A simple question

Next, a simple question might be asked: Which is the probability that the decay of the unstable state occurs between $t = 0$ and the time t? The answer is trivial, since the probability that the decay has actually occurred, denoted as $w(t)$, must be

$$w(t) = 1 - p(t)$$

Similarly, the quantity $h(t) = w'(t) = -p'(t)$ is the probability decay density, with $h(t)dt$ being the probability that the decay occurs between t and $t + dt$.

$$h(t) = w'(t) = -p'(t)$$

A difficult question

Then, a natural, but less easy question is the following: *How to calculate, in a general fashion, the probability, denoted as $w_i(t)$, that the decay occurs in the i-th channel between 0 and t?*

How to calculate the probabilities $w_i(t)$????

N decay channels: formal aspects

$$\Gamma_i(E)$$

$i = 1, ..., N$ enumerates the decay channels

$$E_{th,1} \leq E_{th,2} \leq ... \leq E_{th,N}$$

$$\Pi(E) = \sum_{i=1}^{N} \Pi_i(E) \,, \, \Gamma_i(E) = 2 \, \mathrm{Im} \, \Pi_i(E)$$

partial BW widths are $\Gamma_i = \Gamma_i(M)$

$$\Gamma = \Gamma(M) = \sum_{i=1}^{N} \Gamma_i$$

Breit-Wigner limit

$$d_S(E) = \frac{\Gamma}{2\pi} \frac{1}{(E-M)^2 + \Gamma^2/4}$$

$$\Gamma = \sum_{i=1}^{N} \Gamma_i$$

$$p(t) = e^{-\frac{\Gamma}{\hbar} t}$$

$$w_i(t) = \frac{\Gamma_i}{\Gamma} w(t) = \frac{\Gamma_i}{\Gamma} \left(1 - e^{-\frac{\Gamma}{\hbar} t}\right)$$

$$h_i(t) = \frac{\Gamma_i}{\Gamma} h(t) = \frac{\Gamma_i}{\Gamma} e^{-\frac{\Gamma}{\hbar} t}$$

$$\frac{w_i(t)}{w_j(t)} = \frac{h_i(t)}{h_j(t)} = \frac{\Gamma_i}{\Gamma_j} = const.$$

Partial decay probabilities

$$\Gamma_i(E) = 2g_i^2 \frac{\sqrt{E-E_{th,i}}}{E^2+\Lambda^2}$$

2108.07838 [quant-ph]

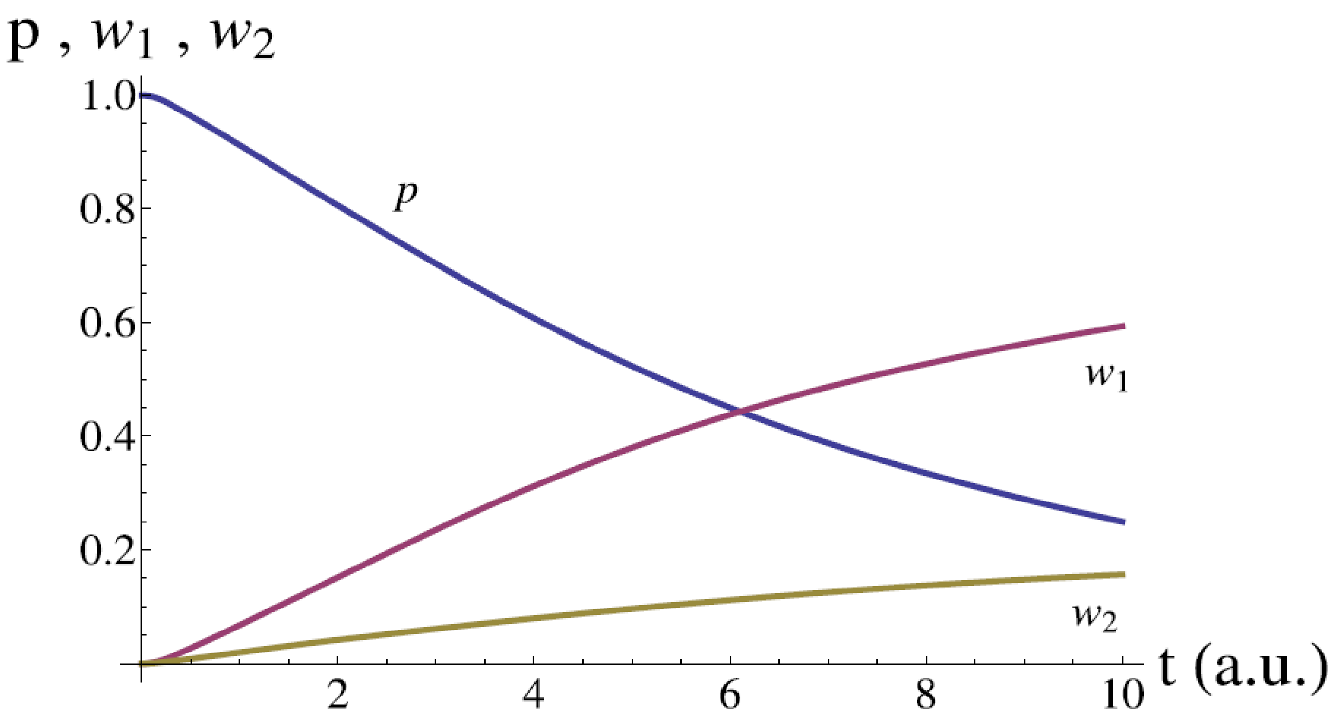


Fig. 1. The survival probability $p(t)$ of Eq. (1) and the decay probabilities $w_1(t)$ and $w_2(t)$ of Eq. (14) are plotted as function of t. The constraint $p+w_1+w_2=1$ holds. Note, t is expressed in a.u. of $[M^-1]$.

$g_1/\sqrt{M}=1$, $g_2/\sqrt{M}=0.6$ $\quad E_{th,1}/M = 1/10$, $E_{th,2}/M = 1/2$, $\Lambda/M = 4$,

Ratio of partial decay probabilities (not a constant)

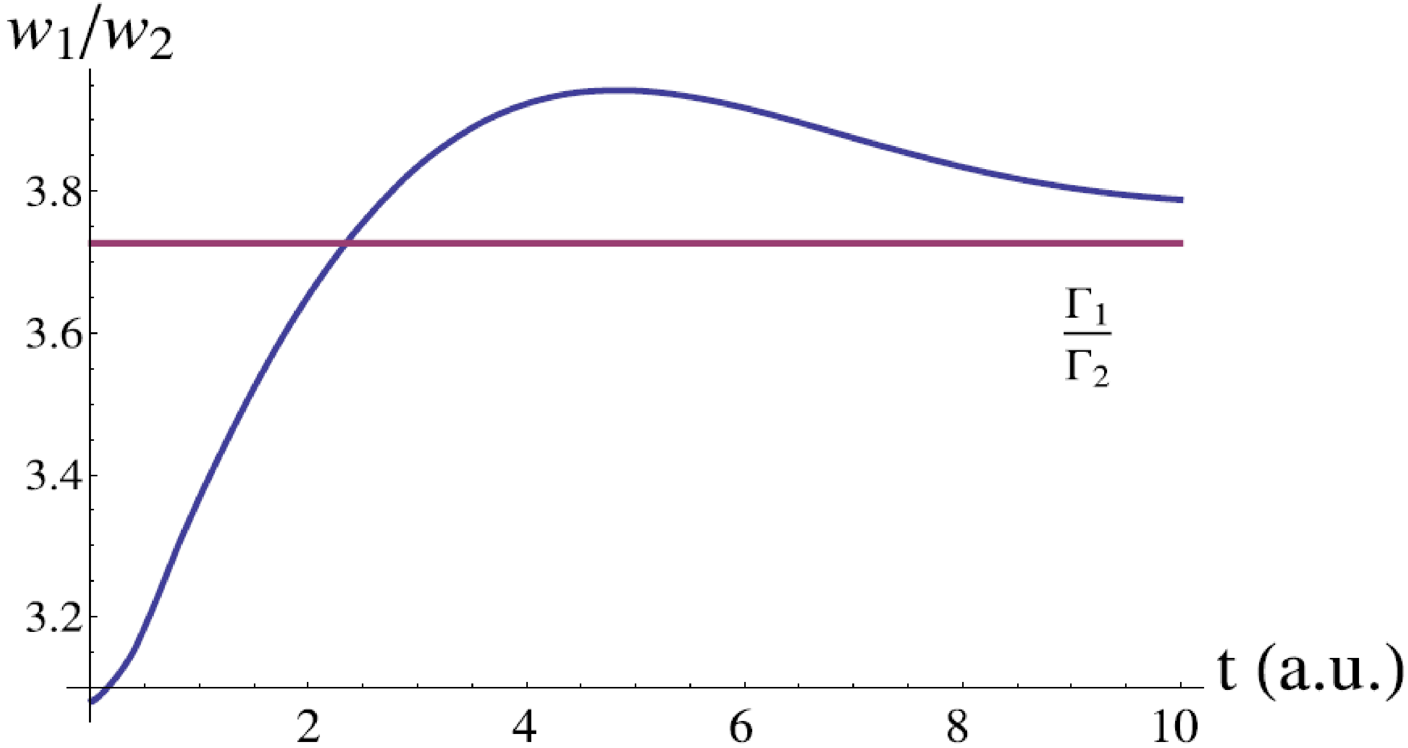


Fig. 2. The ratio w_1/w_2 is plotted as function of t. The straight line corresponds to the BW limit Γ_1/Γ_2, see Eq. (19).

2108.07838 [quant-ph]

Partial probability decay densities

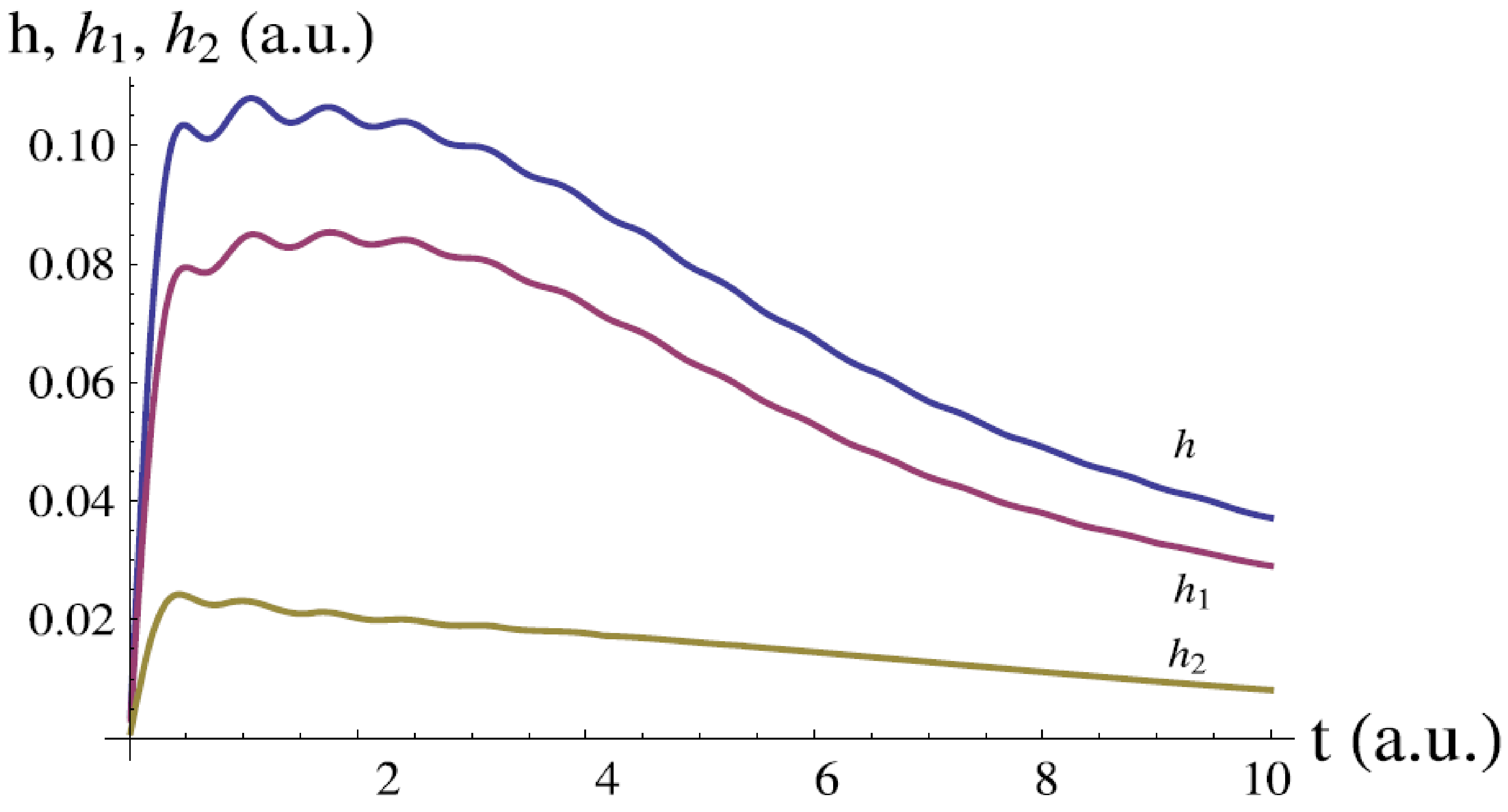


Fig. 3. The quantity $h(t) = w'(t) = -p'(t)$ as well as $h_i(t) = w_i'(t)$ is plotted. The equality $h(t) = h_1(t) + h_2(t)$ holds. Note, h and h_i are in units of $[M]$.

2108.07838 [quant-ph]

Ratio of partial probability decay densities

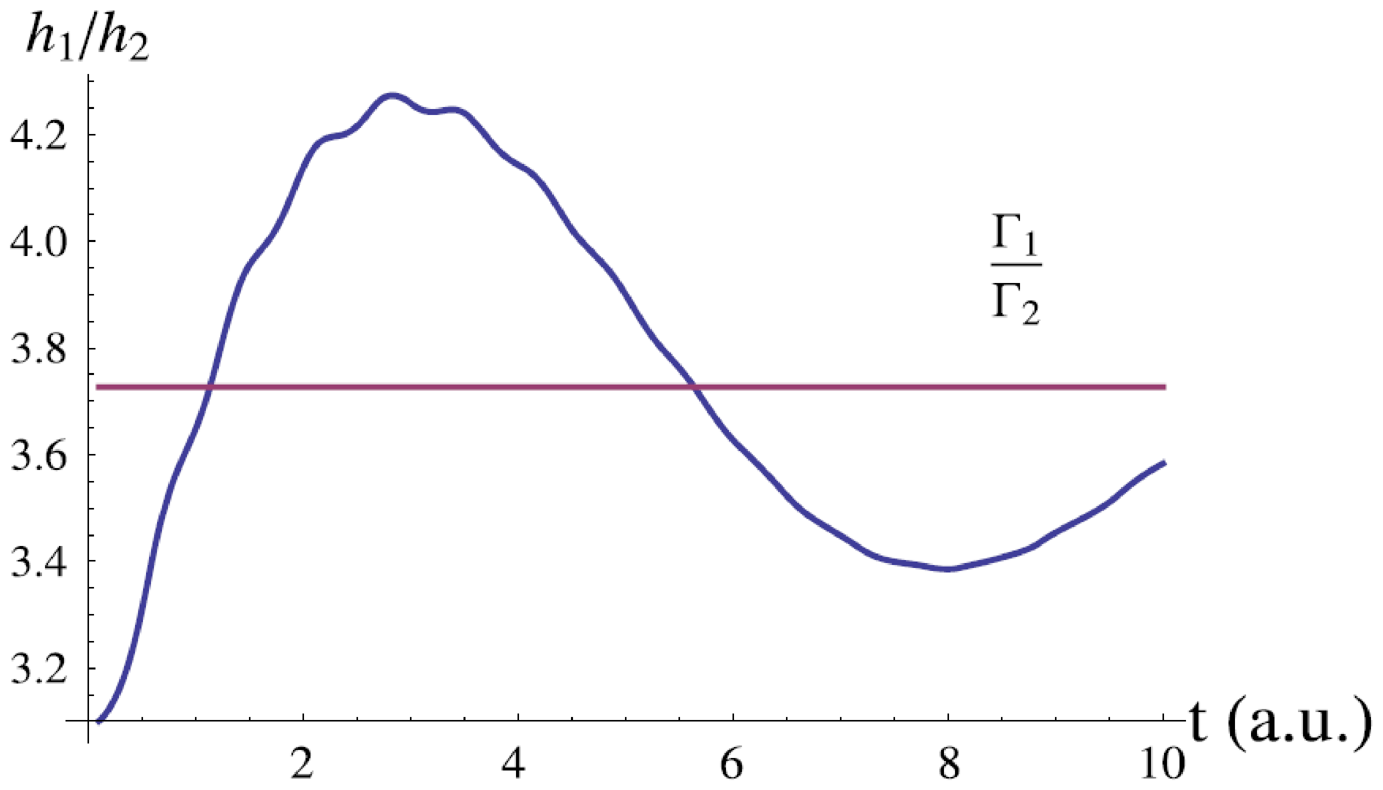
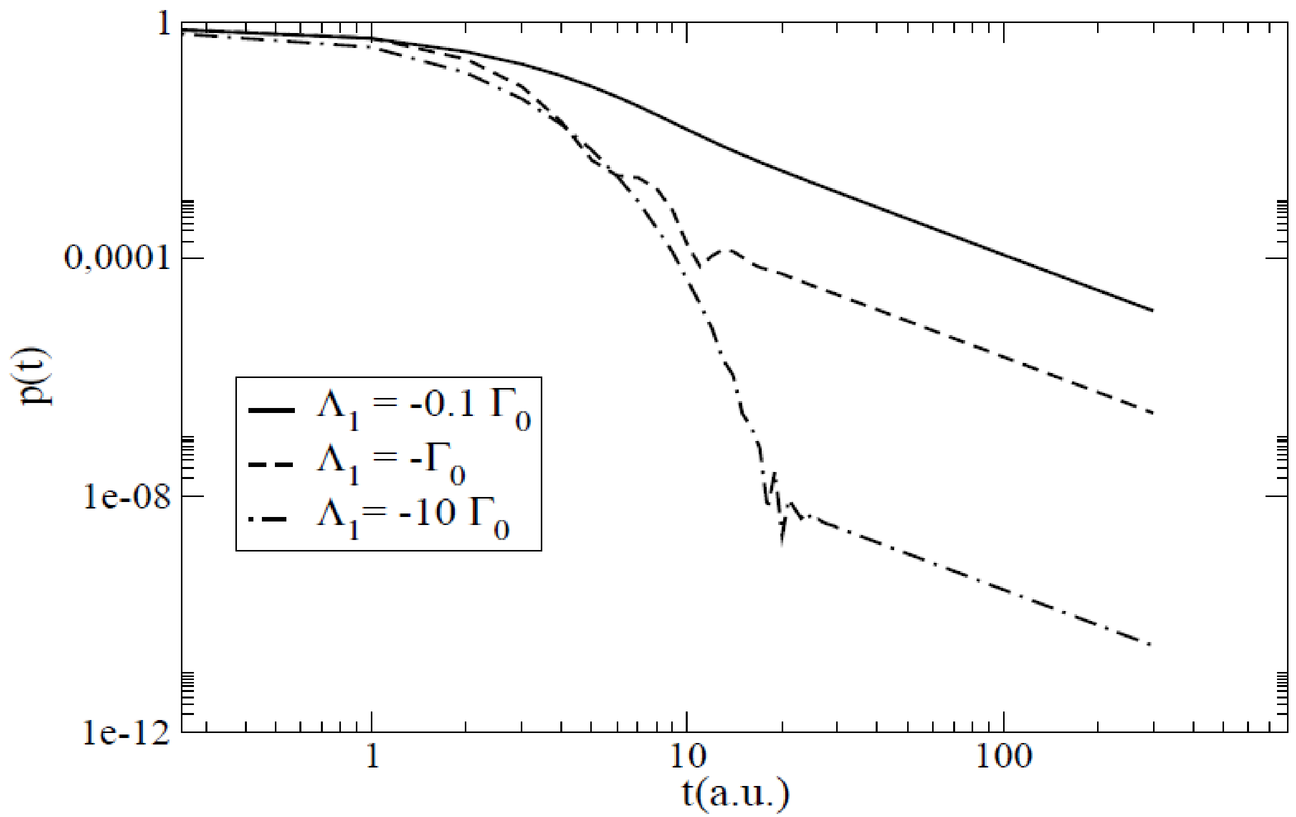


Fig. 4. Ratio h_1/h_2 as function of t. The straight line corresponds to the BW limit Γ_1/Γ_2, see Eq. (19). For the time intervals where $h_1/h_2 > \Gamma_1/\Gamma_2$, the decay in the first channel is enhanced (the opposite is true for $h_1/h_2 < \Gamma_1/\Gamma_2$).

2108.07838 [quant-ph]

Single left-threshold at long times



F.G. and G. Pagliara, [arXiv:1204.1896 [nucl-th]].

Survival amplitude at late times/1

$$z_{pole} = M - i\frac{\Gamma}{2} = M - \frac{i}{2\tau}$$

$$A(t) = -\frac{2i\,\mathrm{Im}[\Pi(z_{pole})]e^{-iz_{pole}t}}{z_{pole} - M + \mathrm{Re}[\Pi(z_{pole})] - i\,\mathrm{Im}[\Pi(z_{pole})]} - \frac{\chi}{M^2}\,t^{-2}$$

$$t_{\text{turn-over}} \simeq 3.03297 \times 10^8\,\mathrm{eV}^{-1} = 1.99634 \times 10^{-7}\,\mathrm{s} \simeq 125.1\,\tau$$