Norway grants

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Testing Bell inequalities in $H \to \tau^+\tau^-$ @ high energy lepton colliders

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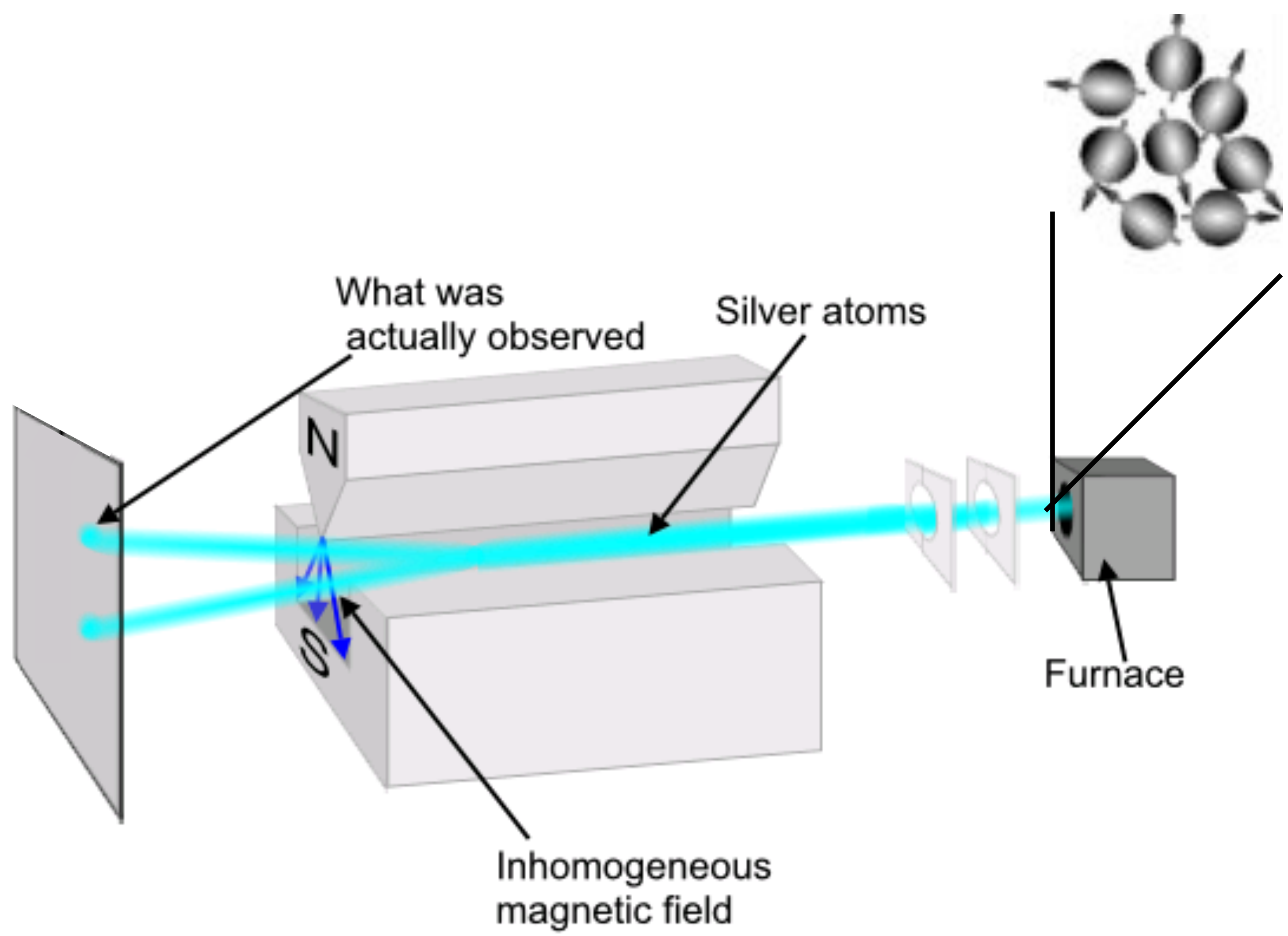
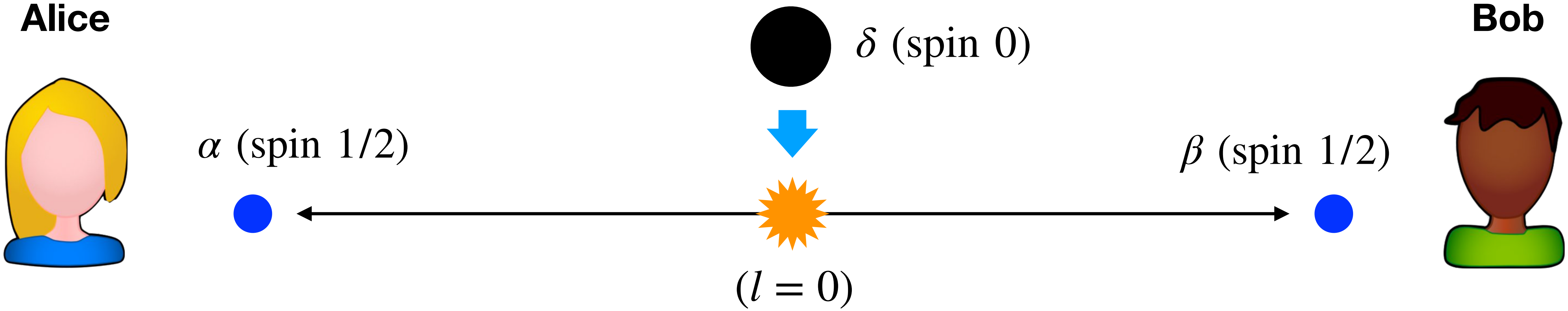
10/10/2022, Seminar @ Jagiellonian University

Spin

In classical mechanics, the components of angular momentum (l_x, l_y, l_z) take continuous real numbers.

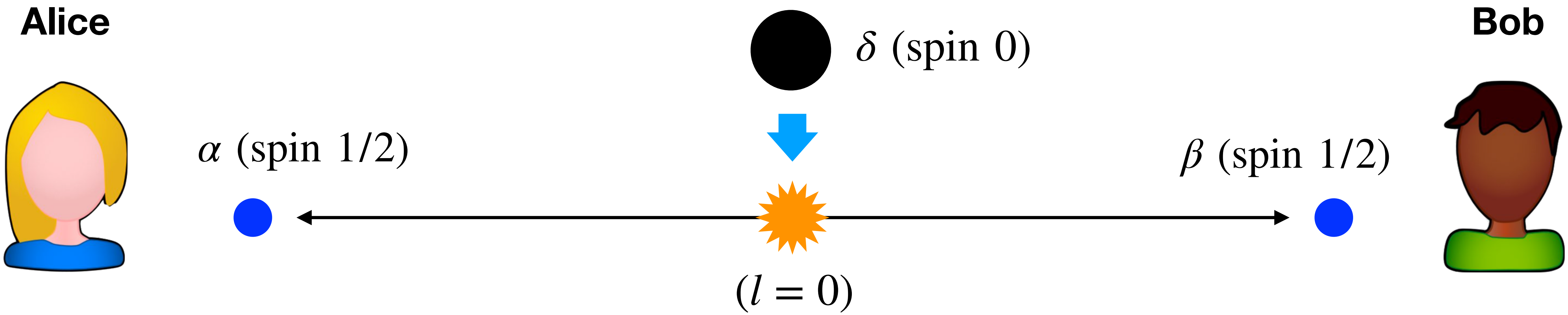
A striking fact, found in the Stern-Gerlach experiment, is that the measurement outcome of spin component is either $+1$ or -1 (in the $\hbar/2$ unit).





- Alice and Bob receive particles *α* and *β*, respectively, and measure the spin *z*-component of their particles. Repeat the process many times.
- Alice and Bob will find their results are completely random (+1 and -1 50-50%)
- Nevertheless, their result is 100% anti-correlated due to the angular momentum conservation. If Alice's result is +1, Bon's result is always -1 and vice versa.

Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+



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Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+
$S_z^\alpha \cdot S_z^\beta$	-	-	-	-	-	-	-	-	-	-	-	-

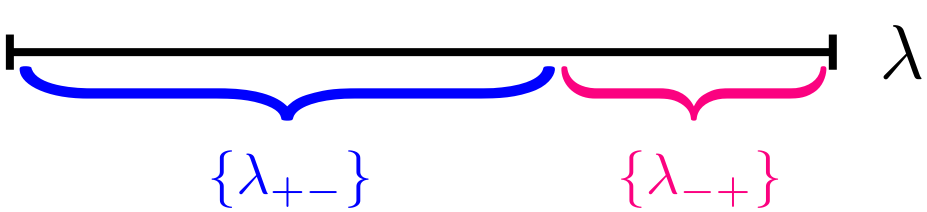
$$\langle S_z^\alpha \cdot S_z^\beta \rangle = -1$$

The most natural explanation would be as follows:

- Since their result is sometimes +1 and sometimes -1, it is natural to think that *the state of α and β are different in each decay.* The result look random, since we don't know in which sate the α and β particles are in each decay.

- This means we can parametrise the state of *α* and *β* by a set of unknown (hidden) variables, λ. For *i*-th decay, their states are:

$$\alpha(\lambda_i), \qquad \beta(\lambda_i)$$



$$\text{If } \lambda_i \in \{\lambda_{+-}\} \implies S_z[\alpha(\lambda_i)] = +1, \;\; S_z[\beta(\lambda_i)] = -1$$

$$\text{If } \lambda_i \in \{\lambda_{-+}\} \implies S_z[\alpha(\lambda_i)] = -1, \;\; S_z[\beta(\lambda_i)] = +1$$

$$P(\lambda \in \{\lambda_{+-}\}) = P(\lambda \in \{\lambda_{-+}\}) = \frac{1}{2}$$

In this explanation:

- Particles have definite properties regardless of the measurement (realism)

- Alice's measurement has no influence on Bob's particle (locality)

The explanation in QM is very different.

Although their outcomes are different in each decay, QM says *the state of the particles are exactly the same for all decays*:

$$|\Psi^{(0,0)}\rangle \underset{\text{up to a phase } e^{i\theta}}{\doteq} \frac{|\overset{\alpha}{+}\overset{\beta}{-}\rangle_z - |-+\rangle_z}{\sqrt{2}}$$

- Before the measurements, particles have no definite spin. Outcomes are undetermined.

(no realism)

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(**no realism**)

- At the moment when Alice makes her measurement, the state collapses into:

$$|\Psi\rangle \xrightarrow[\text{Alice's measurement}]{} \begin{cases} |+,-\rangle_z & \cdots \text{ Alice finds } S_z[\alpha] = +1 \\ |-,+\rangle_z & \cdots \text{ Alice finds } S_z[\alpha] = -1 \end{cases}$$

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Bob's outcome is completely determined (before his measurement) and 100% anti-correlated with Alice's

(non-local)

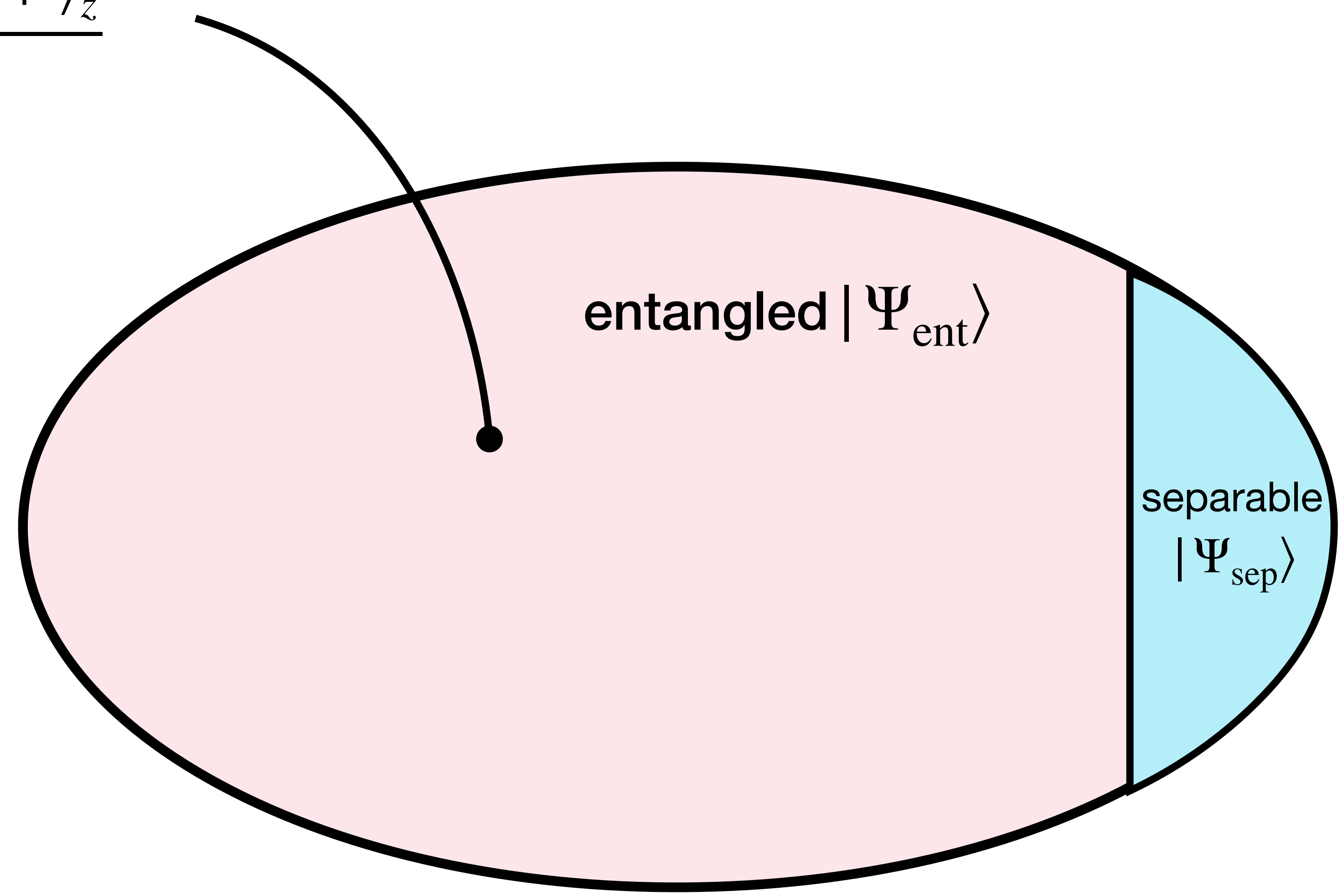
The origin of this bizarre feature is ***entanglement***.

general: $|\Psi\rangle \doteq c_{11}|++\rangle_z + c_{12}|+-\rangle_z + c_{21}|-+\rangle_z + c_{22}|--\rangle_z$

separable: $|\Psi_{\mathrm{sep}}\rangle \doteq \left[c_1^\alpha|+\rangle_z + c_2^\alpha|-\rangle_z\right] \otimes \left[c_1^\beta|+\rangle_z + c_2^\beta|-\rangle_z\right]$

entangled: $|\Psi_{\mathrm{ent}}\rangle \not\doteq \left[c_1^\alpha|+\rangle_z + c_2^\alpha|-\rangle_z\right] \otimes \left[c_1^\beta|+\rangle_z + c_2^\beta|-\rangle_z\right]$

entangled: $|\Psi^{(0,0)}\rangle \doteq \dfrac{|+-\rangle_z - |-+\rangle_z}{\sqrt{2}}$



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Bob's measurement collapses the state of β but doesn't influence the state of α

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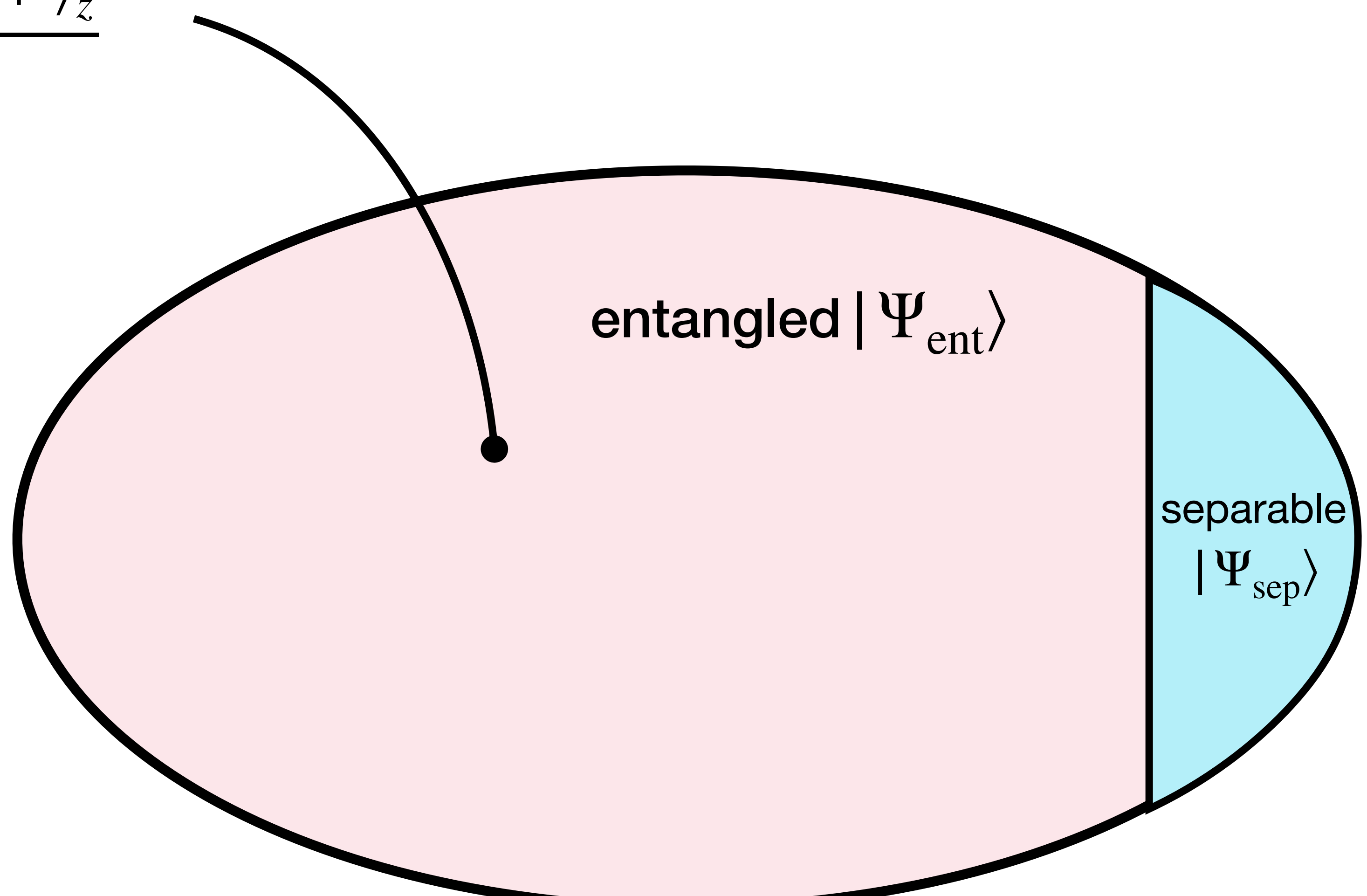

EPR paradox

Einstein, Podolsky and Rosen (EPR) did not like the QM explanation.

EPR's local-real requirement: [Einstein, Podolsky, Rosen 1935]

- Physical observables must be **real**: they have definite values irrespectively with the measurement.

- Physical observables must be **local**: an action in one place cannot influence a physical observable in a space-like separated region.

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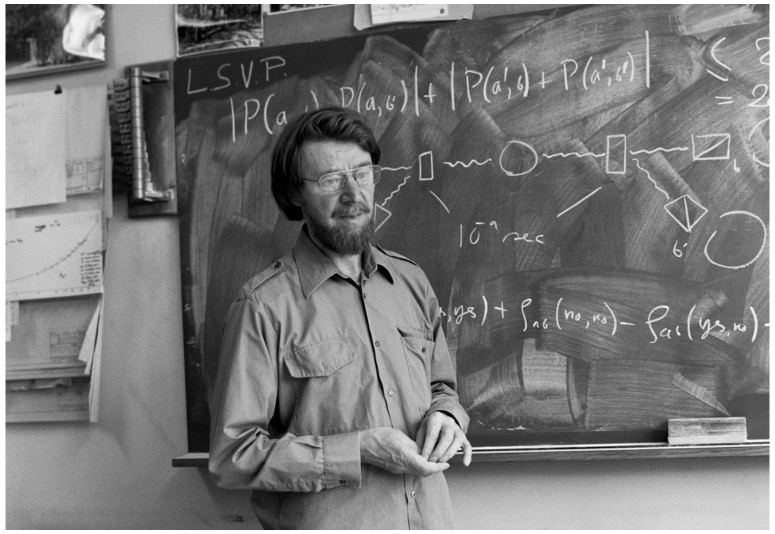
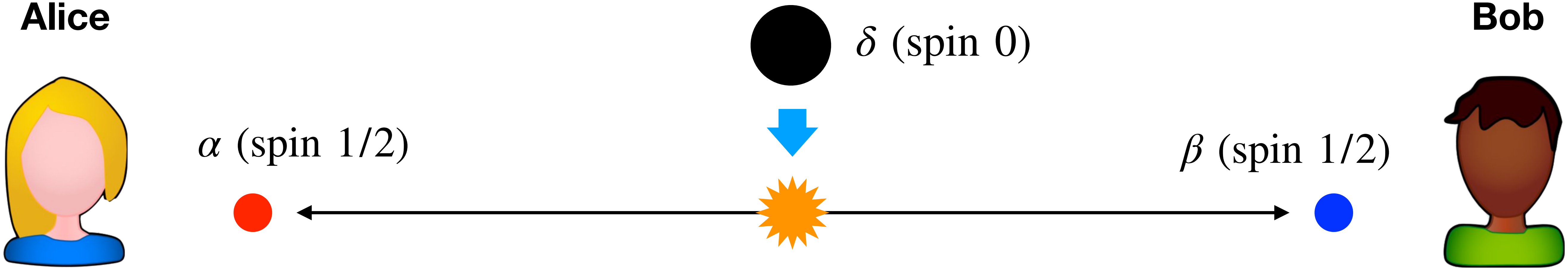
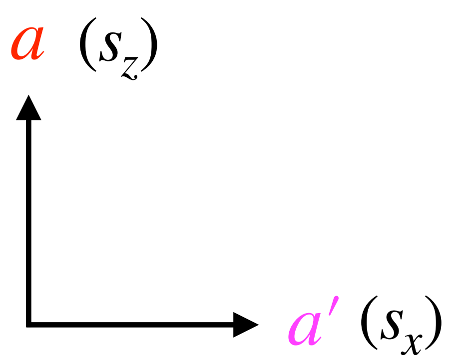
It seems difficult to experimentally discriminate QM and general hidden variable theories.

John Bell (1964) derived simple inequalities that can discriminate QM from any local-real hidden variable theories: **Bell inequalities**

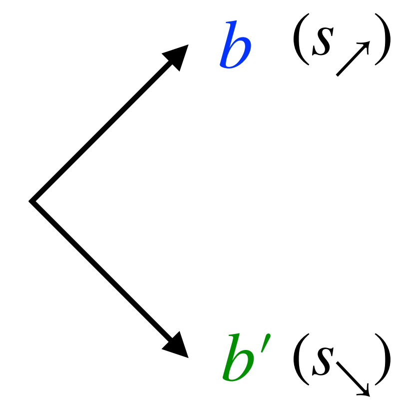






The experiment consists of 4 sessions:

1) Alice and Bob measure $s_a[\alpha]$ and $s_b[\beta]$, respectively. Repeat the measurement many times and calculate $\langle s_a \cdot s_b \rangle$.
2) Repeat (1) but for a and b'.
3) Repeat (1) but for a' and b.
4) Repeat (1) but for a' and b'.



Finally, we construct

$$R_{\text{CHSH}} \equiv \frac{1}{2} \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$$

One can show **in hidden variable theories**:

[Clauser, Horne, Shimony, Holt, 1969]

$$R_{\text{CHSH}} \equiv \frac{1}{2}\left|\langle s_a s_b\rangle - \langle s_a s_{b'}\rangle + \langle s_{a'} s_b\rangle + \langle s_{a'} s_{b'}\rangle\right| \leq 1$$

$$\begin{aligned}
|\langle ab\rangle - \langle ab'\rangle| &= \left|\int d\lambda\,(ab - ab')\,P\right| \\
&= \int d\lambda\,|ab(1 \pm a'b')P - ab'(1 \pm a'b)P| \\
&\leq \int d\lambda\Big(|ab||1 \pm a'b'|P + |ab'||1 \pm a'b|P\Big) \\
&= \int d\lambda\Big[(1 \pm a'b')P + (1 \pm a'b)P\Big] \\
&= 2 \pm (\langle a'b'\rangle + \langle a'b\rangle)
\end{aligned}$$

$$\pm aba'b'P - (\pm aba'b'P) = 0$$

$$a = s_a$$
$$b = s_b$$
$$\vdots$$

$$|ab| = |ab'| = 1$$
$$|1 \pm a'b'|, |1 \pm a'b| \geq 0$$

$$\Rightarrow \quad \tilde{R}_{\text{CHSH}} = \frac{1}{2}\left(|\langle ab\rangle - \langle ab'\rangle| + |\langle a'b\rangle + \langle a'b'\rangle|\right) \leq 1$$

$$\max_{(\vec{a},\vec{b},\vec{a}',\vec{b}')}(R_{\text{CHSH}}) = \max_{(\vec{a},\vec{b},\vec{a}',\vec{b}')}\left(\tilde{R}_{\text{CHSH}}\right)$$

$$\boxed{\langle ab\rangle = \int a(\lambda)b(\lambda)P(\lambda)d\lambda}$$

$$\boxed{\int P(\lambda)d\lambda = 1}$$

In QM, for $|\Psi^{(0,0)}\rangle \doteq \dfrac{|+-\rangle_z - |-+\rangle_z}{\sqrt{2}}$

one can show

$$\langle s_a s_b \rangle = \langle \Psi^{(0,0)} | s_a s_b | \Psi^{(0,0)} \rangle = (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$$

therefore

$$R_{\mathrm{CHSH}} = \frac{1}{2} \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$$

$$= \frac{1}{2} \left| (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}') + (\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}}) + (\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}}') \right|$$

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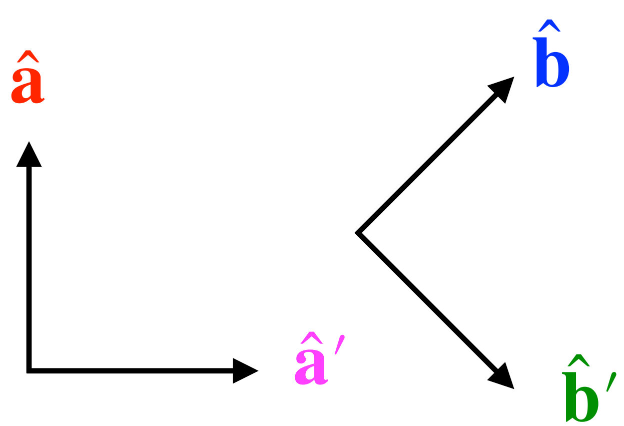
therefore

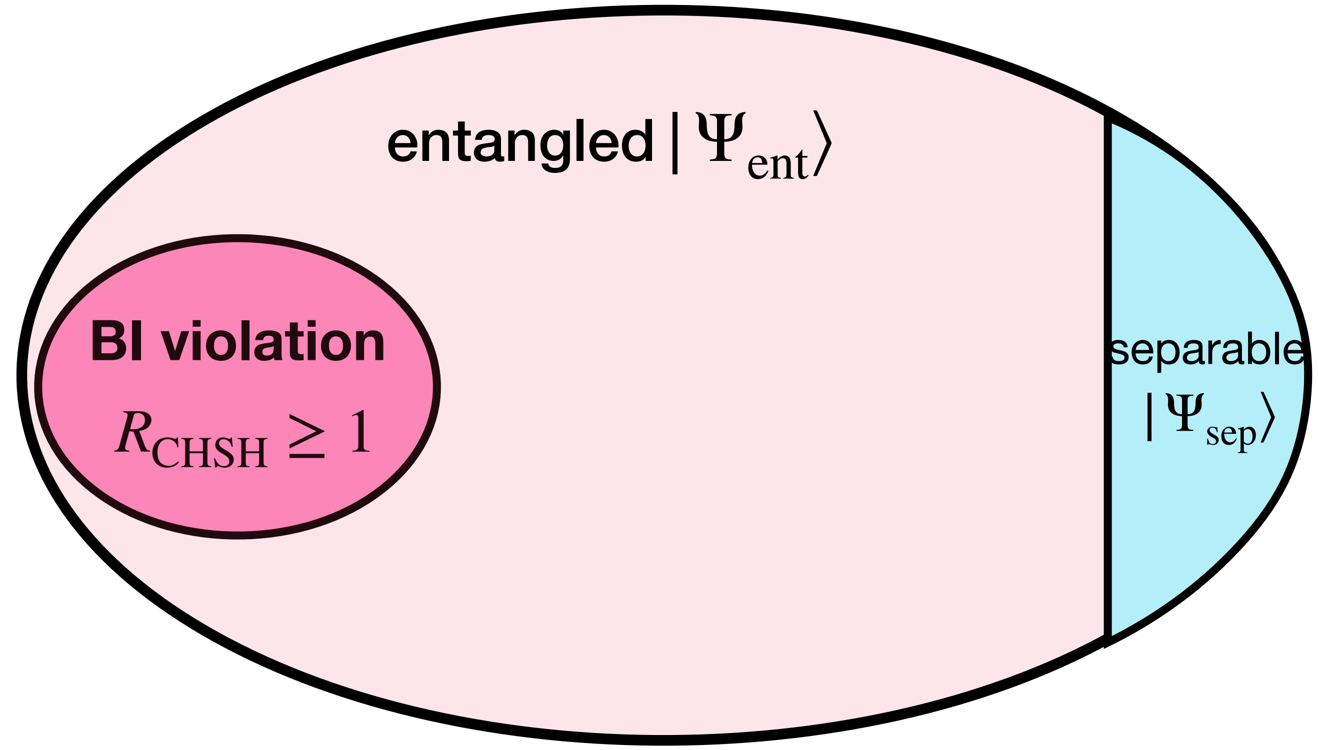
violates the upper bound of hidden variable theories!

$$R_{\text{CHSH}} = \frac{1}{2} \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$$

$$= \frac{1}{2} \left| \underbrace{(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})}_{\frac{1}{\sqrt{2}}} - \underbrace{(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}')}_{-\frac{1}{\sqrt{2}}} + \underbrace{(\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}})}_{\frac{1}{\sqrt{2}}} + \underbrace{(\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}}')}_{\frac{1}{\sqrt{2}}} \right| = \sqrt{2}$$



$$R_{\text{CHSH}} \leq \begin{cases} 1 & (\text{HV theories}) \\ \sqrt{2} & (\text{QM}) \end{cases}$$



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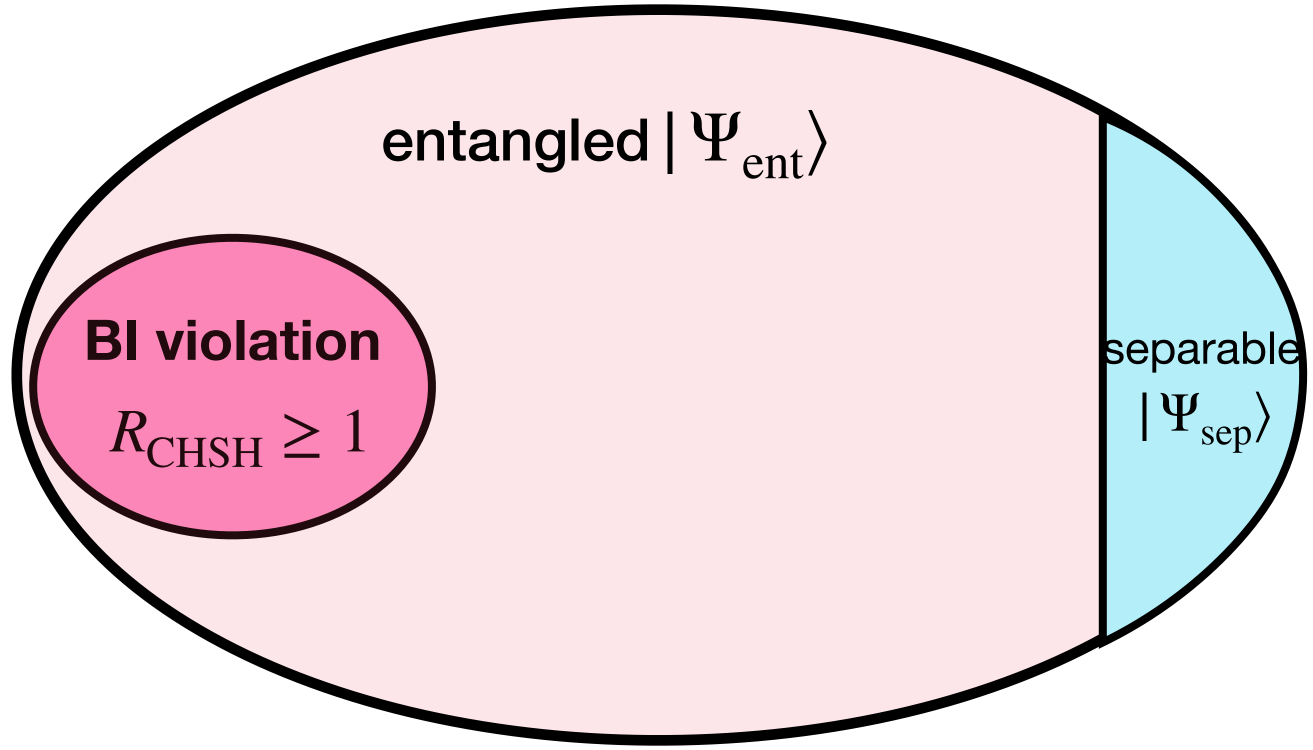

✤ Violation of Bell inequalities has been observed in low energy experiments:

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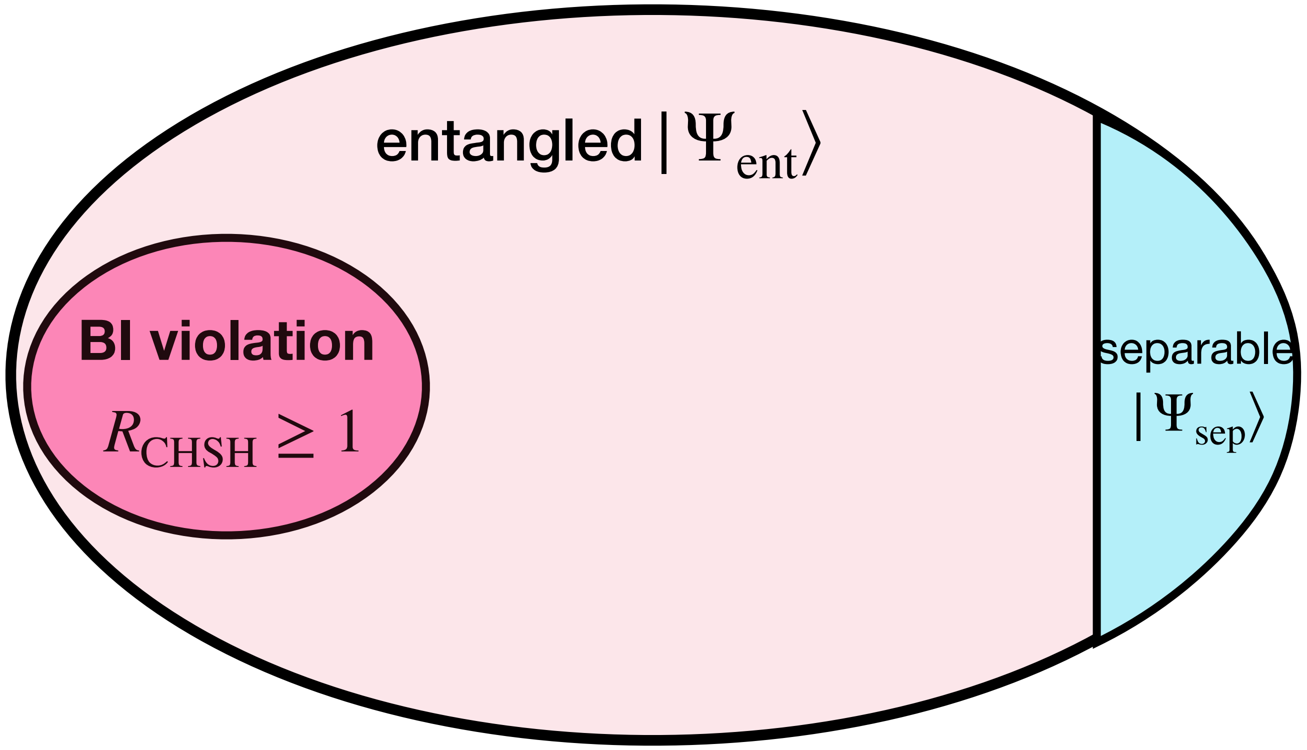

✤ Violation of Bell inequalities has been observed in low energy experiments:

- **Entangled photon pairs** (from decays of Calcium atoms)

  Crauser, Horne, Shimony, Holt (1969), Freedman and Clauser (1972), A. Aspect et. al. (1981, 1982), Y. H. Shih, C. O. Alley (1988), L. K. Shalm et al. (2015) [5σ]

- **Entangled proton pairs** (from decays of ^{2}He)

  M. M. Lamehi-Rachti, W. Mitting (1972), H. Sakai (2006)

- $K^0\overline{K^0}, B^0\overline{B^0}$ **flavour oscillation** CPLEAR (1999), Belle (2004, 2007)

Bell inequality and entanglement have not been tested at high energy regime E ~ TeV

Can we test Bell inequality and entanglement at high energy colliders?

- Entanglement in $pp \to t\bar{t}$ @ LHC — Y. Afik, J. R. M. de Nova (2020)

- Bell inequality test in $pp \to t\bar{t}$ @ LHC — M. Fabbrichesi, R. Floreanini, G. Panizzo (2021); C. Severi, C. D. Boschi, F. Maltoni, M. Sioli (2021); J. A. Aguilar-Saavedra, J. A. Casas (2022)

- Bell inequality test in $H \to WW^*$ @ LHC — A. J. Barr (2021)

- Quantum property test in $H \to \tau^+\tau^-$ @ high energy e^+e^- colliders ← **this talk**

Density operator

probability of having $|\Psi_1\rangle$

- For a statistical ensemble $\big\{\{p_1 : |\Psi_1\rangle\},\ \{p_2 : |\Psi_2\rangle\},\ \{p_3 : |\Psi_3\rangle\},\ \cdots\big\}$, we define the **density operator/matrix**

$$\hat{\rho} \equiv \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \qquad \rho_{ab} \equiv \langle e_a|\hat{\rho}|e_b\rangle$$

$$0 \le p_k \le 1$$

$$\sum_k p_k = 1$$

$$\langle e_a|e_b\rangle = \delta_{ab}$$

- Density matrices satisfy the conditions:
  - $\hat{\rho}^\dagger = \hat{\rho}$
  - $\mathrm{Tr}\,\hat{\rho} = 1$
  - $\hat{\rho}$ is positive definite, that is ${}^\forall|\varphi\rangle;\ \langle\varphi|\hat{\rho}|\varphi\rangle \ge 0.$

- The expectation of an observable $\hat{O}$ is calculated by

$$\langle\hat{O}\rangle = \mathrm{Tr}\left[\hat{O}\hat{\rho}\right]$$

Spin 1/2 biparticle system

- The spin system of α and β particles has 4 independent bases:

$$\left(\, |e_1\rangle, \, |e_2\rangle, \, |e_3\rangle, \, |e_4\rangle \, \right) \; = \; \left(\, |++\rangle, \, |+-\rangle, \, |-+\rangle, \, |--\rangle \, \right)$$

- ==> ρ_{ab} is a 4 x 4 matrix (hermitian, Tr=1). It can be expanded as

3 x 3 matrix

$$\rho = \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} + B_i \cdot \sigma_i \otimes \mathbf{1} + \overline{B}_i \cdot \mathbf{1} \otimes \sigma_i + C_{ij} \cdot \sigma_i \otimes \sigma_j\right) \qquad B_i, \overline{B}_i, C_{ij} \in \mathbb{R}$$

- For the spin operators $\hat{s}^\alpha$ and $\hat{s}^\beta$,

spin-spin correlation

$$\langle \hat{s}_i^\alpha \rangle = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{\rho}\right] = B_i \qquad \langle \hat{s}_i^\beta \rangle = \mathrm{Tr}\left[\hat{s}_i^\beta \hat{\rho}\right] = \overline{B}_i \qquad \boxed{\langle \hat{s}_i^\alpha \hat{s}_j^\beta \rangle = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{s}_j^\beta \hat{\rho}\right] = C_{ij}}$$

$H \to \tau^+\tau^-$

$$\mathscr{L}_{\rm int} = -\frac{m_\tau}{v_{\rm SM}}\kappa H\,\bar\psi_\tau(\cos\delta + i\gamma_5\sin\delta)\,\psi_\tau$$

SM: $(\kappa, \delta) = (1,0)$

The density matrix can be computed from the matrix elements:

$$\rho_{mn,\bar m\bar n} = \frac{\mathcal{M}^{*n\bar n}\mathcal{M}^{m\bar m}}{\sum_{m\bar m}|\mathcal{M}^{m\bar m}|^2}$$

$$\mathcal{M}^{m\bar m} = c\,\bar u^m(p)(\cos\delta + i\gamma_5\sin\delta)v^{\bar m}(\bar p)$$

$$\rho_{mn,\bar m\bar n} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & e^{-i2\delta} & 0 \\ 0 & e^{i2\delta} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_i = \overline{B}_i = 0$$

$$C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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$$|\Psi_{H\to\tau\tau}(\delta)\rangle \propto |+-\rangle + e^{i2\delta}|-+\rangle$$

$\delta = 0$ (CP even)

$$|\Psi^{(1,m)}\rangle \propto \begin{pmatrix} |++\rangle \\ \boxed{|+-\rangle + |-+\rangle} \\ |--\rangle \end{pmatrix}$$

$\delta = \pi/2$ (CP odd)

$$|\Psi^{(0,0)}\rangle \propto \boxed{|+-\rangle - |-+\rangle}$$

Parity: $P = (\eta_f \eta_{\bar{f}}) \cdot (-1)^l$ with $\eta_f \eta_{\bar{f}} = -1$:

$$J^P = \begin{cases} 0^+ \implies -l = s = 1 \\ 0^- \implies \quad l = s = 0 \end{cases}$$

Entanglement

- If the state is separable (not entangled),

$$\rho = \sum_k p_k \rho_k^\alpha \otimes \rho_k^\beta$$

$$0 \leq p_k \leq 1$$

$$\sum_k p_k = 1$$

then, a modified matrix by the partial transpose

$$\rho^{T_\beta} \equiv \sum_k p_k \rho_k^\alpha \otimes [\rho_k^\beta]^T$$

is also a physical density matrix, i.e. Tr=1 and non-negative.

- For biparticle systems, entanglement $\Longleftrightarrow$ ρ^{T_β} to be non-positive.

Peres-Horodecki (1996, 1997)

- A simple sufficient condition for entanglement is:

$$E \equiv C_{11} + C_{22} - C_{33} > 1$$

$$(E = 2\cos 2\delta + 1 \quad \text{for} \quad H \to \tau^+\tau^-)$$

$$(E = 3 \quad \text{(maximally entangled)} \quad \text{for} \quad H \to \tau^+\tau^- \text{ in SM})$$

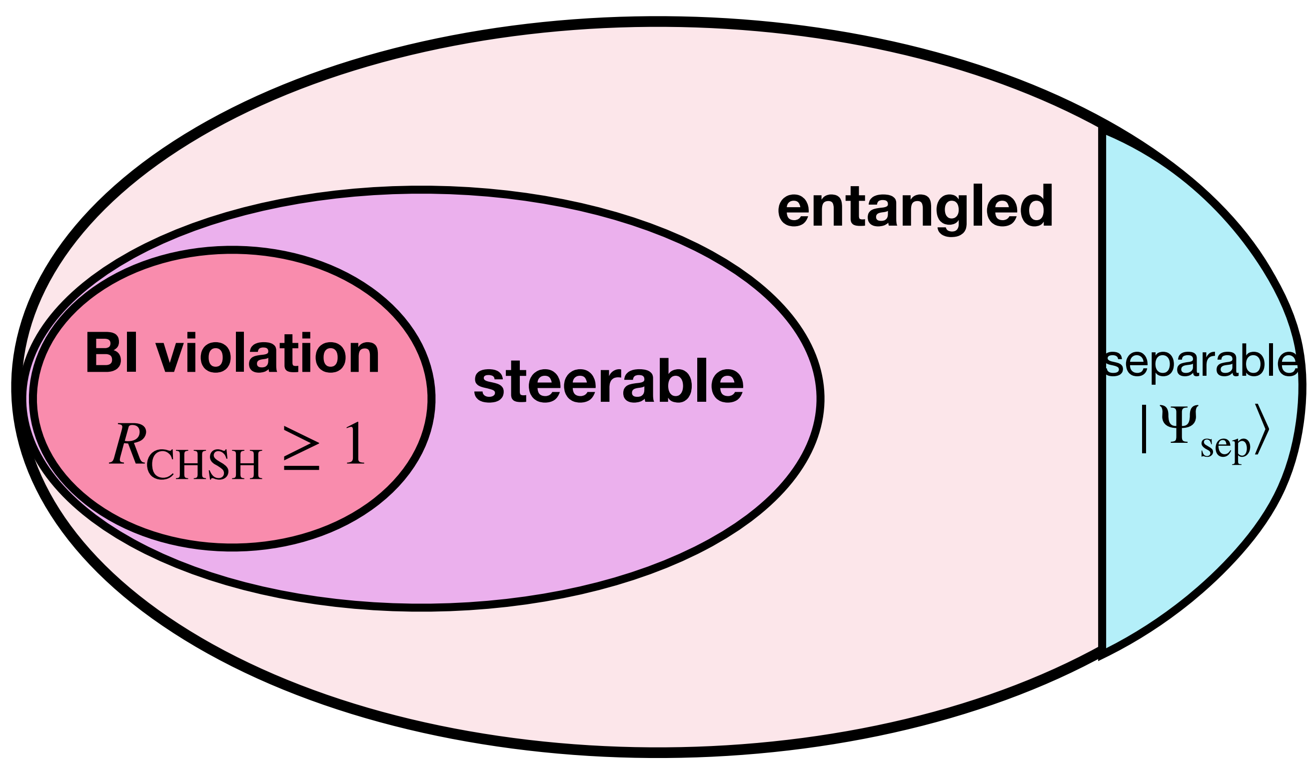
Steering

[Schrödinger 1935]

- Steering for Alice is Alice's ability to "steer" Bob's local state by her measurement.
- Suppose Alice and Bob measure the observables $\mathscr{A}$ and $\mathscr{B}$, and obtained the outcomes a and b. The state is said to be *steerable* by Alice, if it is ***not*** possible to write this probability in a form: [Jones, Wiseman, Doherty 2007]

Bob's local state

$$p(a,b) = \sum_{\lambda} p(a\,|\,\lambda) \cdot p_Q(b\,|\,\lambda) \qquad p_Q(b\,|\,\lambda) = \mathrm{Tr}\left[\rho_B(\lambda)\,|\,b\rangle\langle b\,|\,\right]$$



Steering

- For unpolarised cases, $\langle \hat{s}_i^A \rangle = \langle \hat{s}_i^B \rangle = 0$, a necessary and sufficient condition for steerability is given by: [Jevtic, Hall, Anderson, Zwierz, Wiseman 2015]

$$\mathcal{S}[\rho] \equiv \frac{1}{2\pi} \int d\Omega_{\mathbf{n}} \sqrt{\mathbf{n}^T C^T C \mathbf{n}} \qquad\qquad \mathcal{S}[\rho] > 1$$

- In $H \to \tau^+ \tau^-$,

$$C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow C^T C = \mathbf{1} \Rightarrow \mathcal{S}[\rho] = 2 \quad \text{(independent of } \delta \text{)}$$

- Let's suppose a spin 1/2 particle α is ***at rest*** and spinning in the $\mathbf{s}$ direction.
- α decays into a measurable particle l_α and the rest X $\quad \alpha \to l_\alpha + (X)$
- The decay distribution is generally given by

$$\frac{d\Gamma}{d\Omega} \propto 1 + x_\alpha (\hat{\mathbf{l}}_\alpha \cdot \mathbf{s})$$

$\hat{\mathbf{l}}_\alpha$ is a unit direction vector of l_α, measured at the rest frame of α

- $x \in [-1,\ 1]$ is called *spin-analysing power and* depends on the decay.

$$\tau^- \to \pi^- + (\nu_\tau) \quad \Longrightarrow \quad x = 1$$

- One can show for $\alpha + \beta \ \to \ [l_\alpha + (X)] \ + \ [l_\beta + X]$ and $\xi_{ij} \equiv (\hat{\mathbf{l}}_\alpha)_i (\hat{\mathbf{l}}_\beta)_j$

$$\frac{d\sigma}{d\xi_{ij}} = \left(1 - C_{ij}\right) \cdot \ln\left(\frac{1}{\xi_{ij}}\right)$$

$$\Rightarrow \quad C_{ij} = 4 \cdot \frac{N(\xi_{ij} > 0) - N(\xi_{ij} < 0)}{N(\xi_{ij} > 0) + N(\xi_{ij} < 0)}$$

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2}\left|\langle s_a s_b\rangle - \langle s_a s_{b'}\rangle + \langle s_{a'} s_b\rangle + \langle s_{a'} s_{b'}\rangle\right|$$

$$= \frac{9}{2\,|x_\alpha x_\beta|}\left|\left\langle(\hat{\mathbf{l}}_\alpha)_a(\hat{\mathbf{l}}_\beta)_b\right\rangle - \left\langle(\hat{\mathbf{l}}_a)(\hat{\mathbf{l}}_\beta)_{b'}\right\rangle + \left\langle(\hat{\mathbf{l}}_\alpha)_{a'}(\hat{\mathbf{l}}_\beta)_b\right\rangle + \left\langle(\hat{\mathbf{l}}_\alpha)_{a'}(\hat{\mathbf{l}}_\beta)_{b'}\right\rangle\right|$$

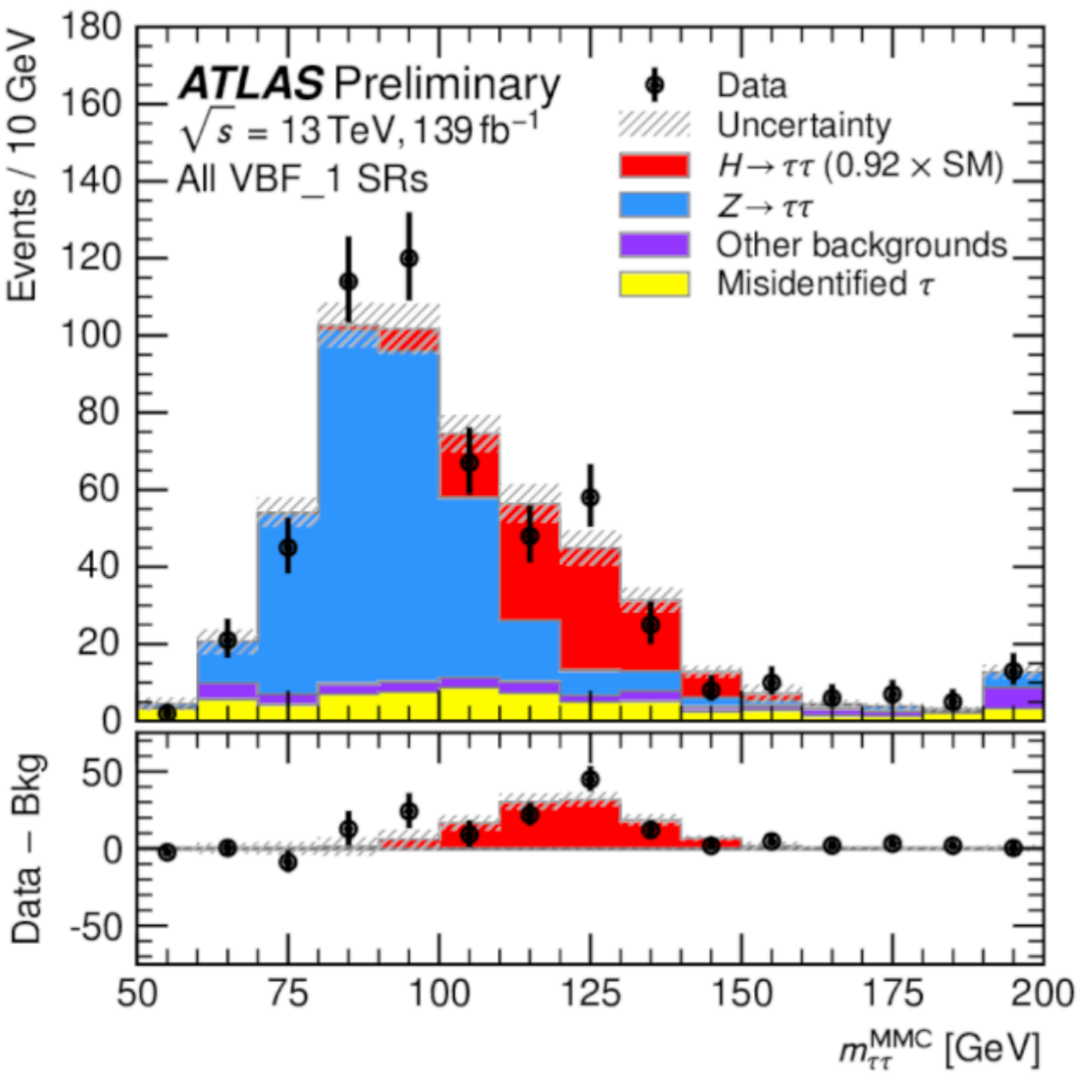
R_{CHSH} can be directly calculated

once the unit vectors $(\hat{\mathbf{a}}, \hat{\mathbf{a}}', \hat{\mathbf{b}}, \hat{\mathbf{b}}')$ are fixed.

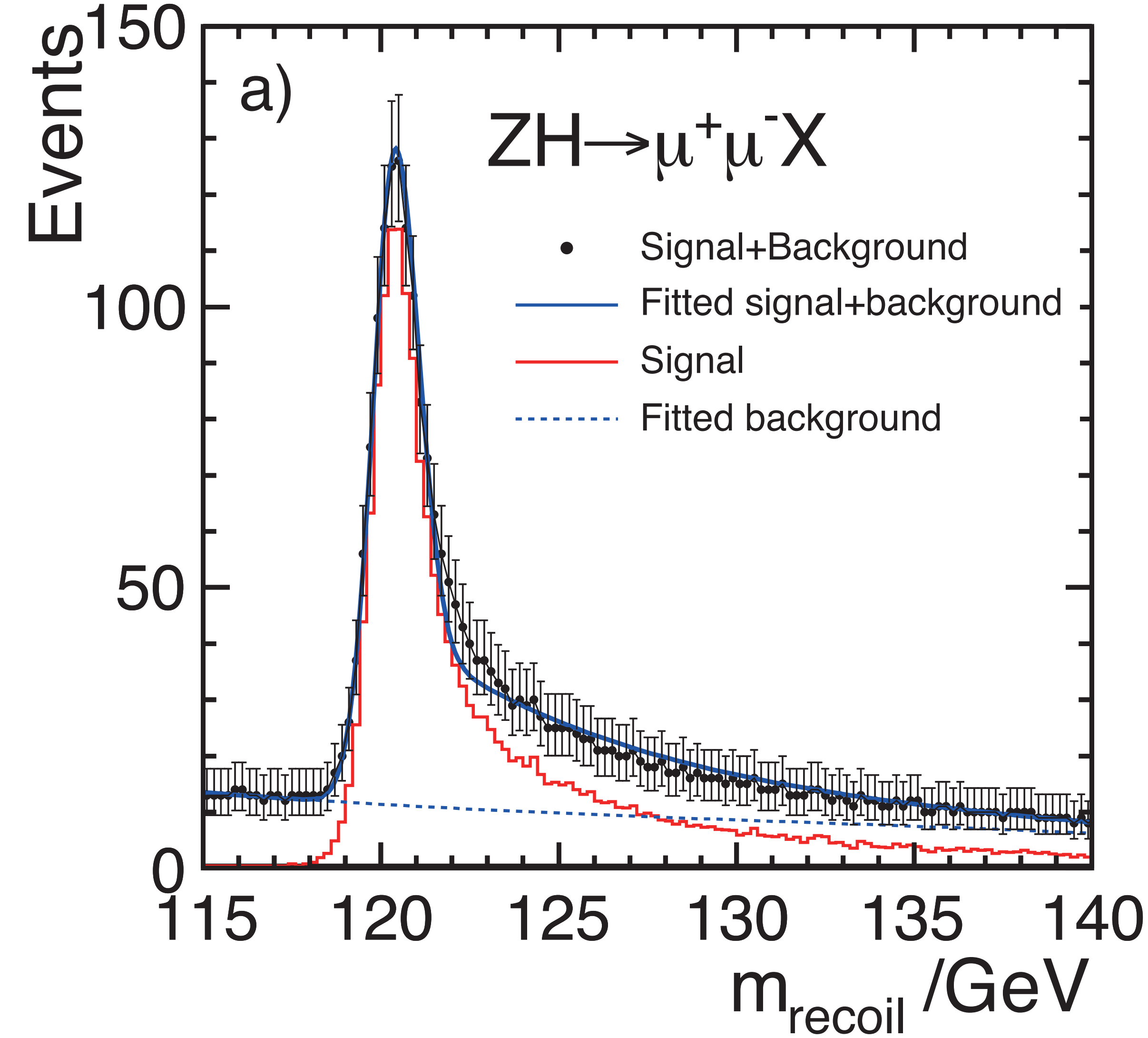
$H \to \tau^+\tau^-$ @ lepton colliders

- Background $Z/\gamma \to \tau^+\tau^-$ is much smaller for lepton colliders
- We need to reconstruct each τ rest frame to measure $\hat{\mathbf{l}}$. This is challenging at hadron colliders since partonic CoM energy is unknown for each event

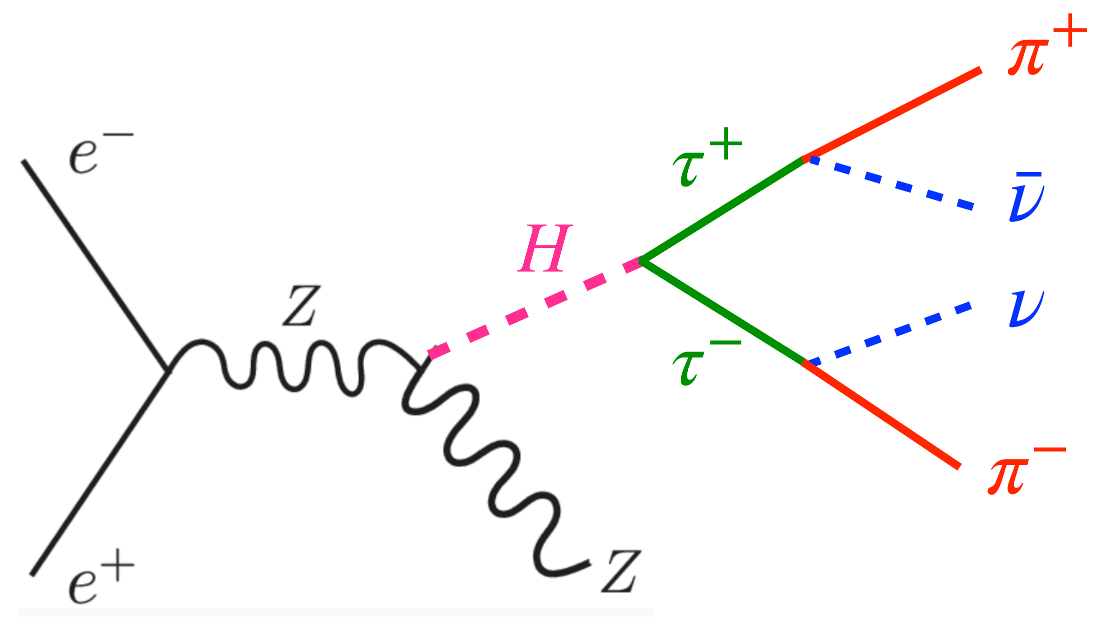
LHC



ILC



- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^{\nu}, p_y^{\nu}, p_z^{\nu}), \ (p_x^{\bar{\nu}}, p_y^{\bar{\nu}}, p_z^{\bar{\nu}})$.



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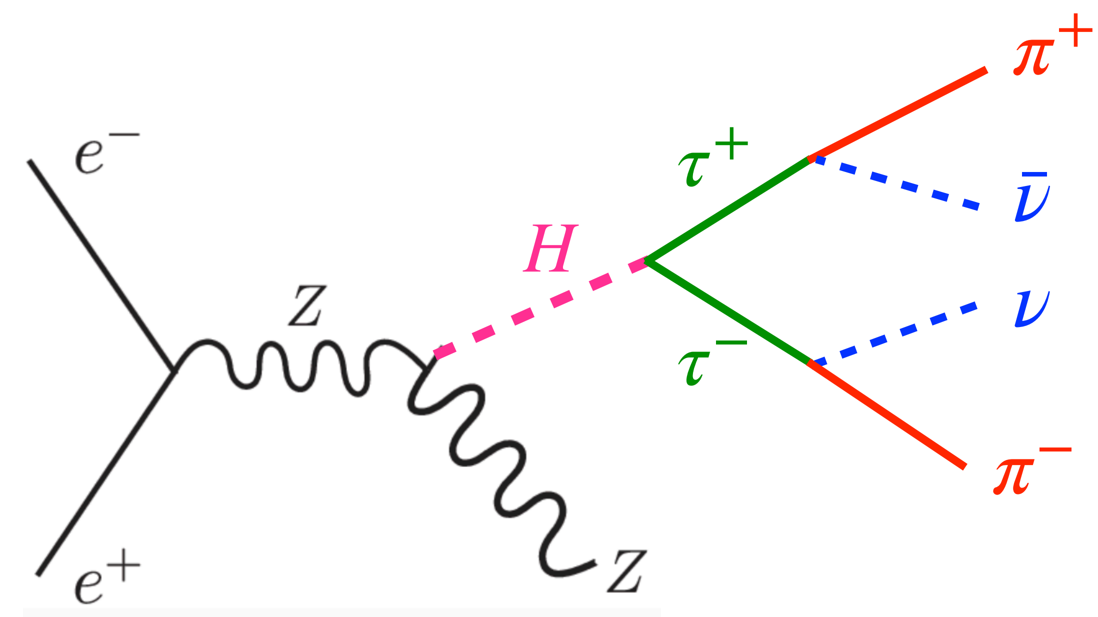
- **6** unknowns can be constrained by **2** mass-shell conditions and **4** energy-momentum conservation.



$$m_\tau^2 = (p_{\tau^+})^2 = (p_{\pi^+} + p_{\bar{\nu}})^2$$

$$m_\tau^2 = (p_{\tau^-})^2 = (p_{\pi^-} + p_{\nu})^2$$

$$(p_{ee} - p_Z)^\mu = p_H^\mu = \left[(p_{\pi^-} + p_\nu) + (p_{\pi^+} + p_{\bar{\nu}})\right]^\mu$$

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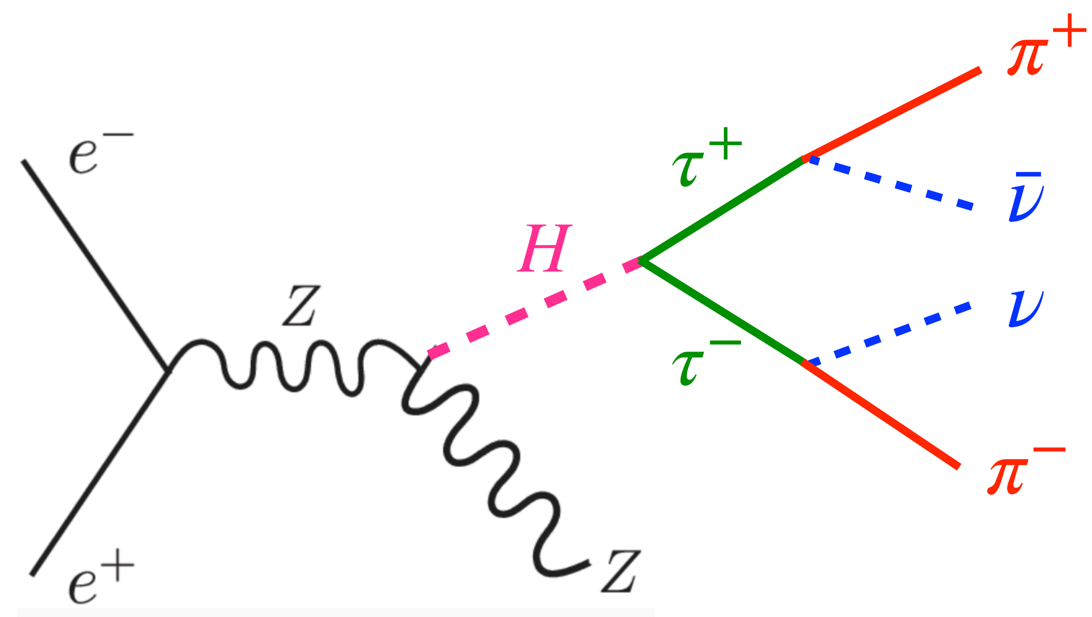
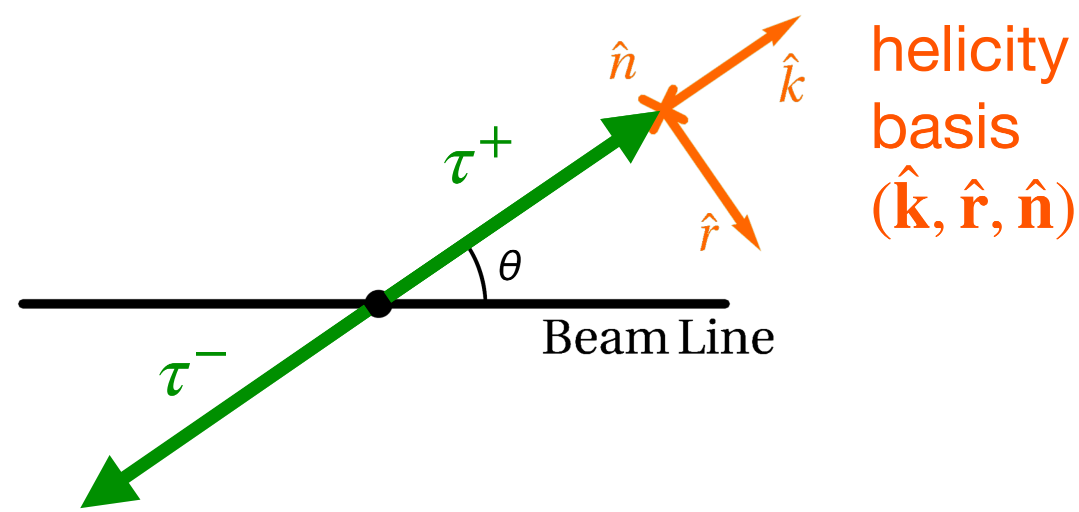
- With the reconstructed momenta, we define $(\hat{\mathbf{k}}, \hat{\mathbf{r}}, \hat{\mathbf{n}})$ basis at the Higgs rest frame.





- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^\nu, p_y^\nu, p_z^\nu)$, $(p_x^{\bar\nu}, p_y^{\bar\nu}, p_z^{\bar\nu})$.

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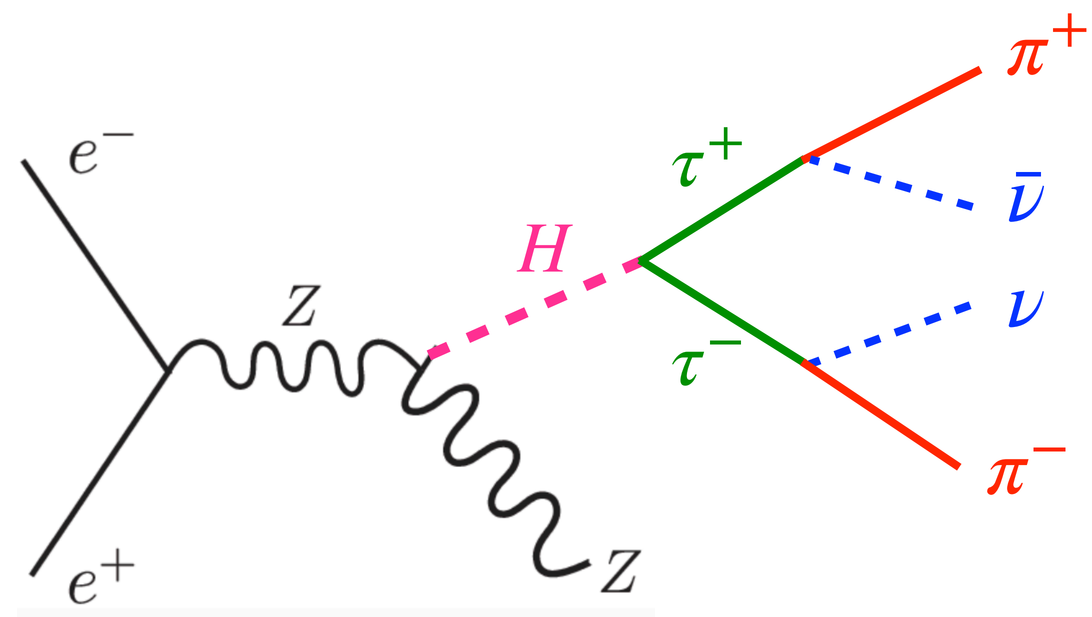
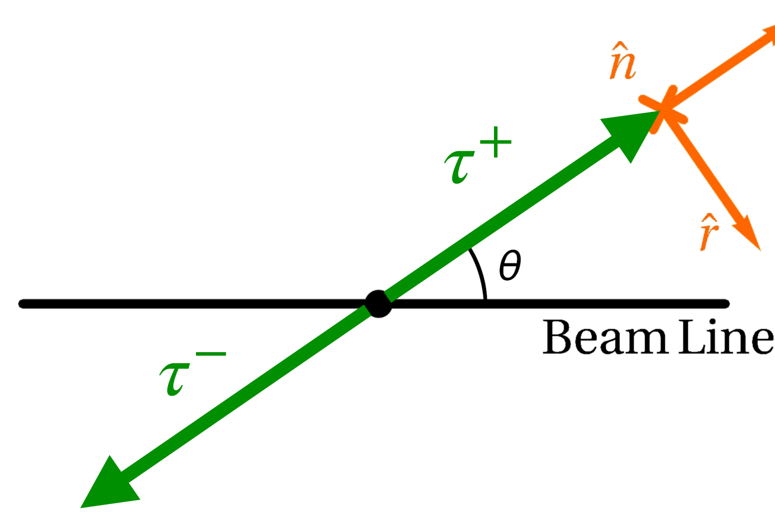
- With the reconstructed momenta, we define $(\hat{\mathbf{k}}, \hat{\mathbf{r}}, \hat{\mathbf{n}})$ basis at the Higgs rest frame.

- In the $\tau^{+(-)}$ rest frame, we measure the direction of $\pi^{+(-)}$, $\hat{\mathbf{l}}^+$ and $\hat{\mathbf{l}}^-$, and calculate R_{CHSH} directly with

$$(\hat{\mathbf{a}}, \hat{\mathbf{a}}', \hat{\mathbf{b}}, \hat{\mathbf{b}}') = \left(\hat{\mathbf{k}}, \hat{\mathbf{r}}, \frac{1}{\sqrt{2}}(\hat{\mathbf{k}} + \hat{\mathbf{r}}), \frac{1}{\sqrt{2}}(\hat{\mathbf{k}} - \hat{\mathbf{r}})\right)$$

and measure C_{ij} from FB asymmetry.





$$C_{ij} = 4 \cdot \frac{N(\hat{\mathbf{l}}_i^+ \hat{\mathbf{l}}_j^+ > 0) - N(\hat{\mathbf{l}}_i^+ \hat{\mathbf{l}}_j^+ < 0)}{N(\hat{\mathbf{l}}_i^+ \hat{\mathbf{l}}_j^+ > 0) + N(\hat{\mathbf{l}}_i^+ \hat{\mathbf{l}}_j^+ < 0)}$$

$$\mathcal{S}[\rho] \equiv \frac{1}{2\pi} \int d\Omega_{\mathbf{n}} \sqrt{\mathbf{n}^T C^T C \mathbf{n}}$$

Simulation

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$
$\sigma(e^+e^- \to HZ)$ (fb)	240.1	240.3
# of signal ($\sigma \cdot \mathrm{BR} \cdot L$)	414	691

- Generate the SM events $(\kappa, \delta) = (1{,}0)$ with **MadGraph5**.

- We incorporate the detector effect by **smearing energies** of visible particles with

$$E^{\mathrm{true}} \to E^{\mathrm{obs}} = (1 + \sigma_E \cdot \omega) \cdot E^{\mathrm{true}} \qquad \sigma_E = 0.03$$

random number from the normal distribution

- We perform **100 pseudo-experiments** to estimate the statistical uncertainties of the measurements.

Results

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} -0.592 \pm 0.149 & -0.008 \pm 0.137 & 0.0151 \pm 0.176 \\ -0.0151 \pm 0.142 & -0.554 \pm 0.159 & 0.002 \pm 0.180 \\ 0.006 \pm 0.169 & 0.003 \pm 0.160 & 0.423 \pm 0.172 \end{pmatrix}$	$\begin{pmatrix} -0.369 \pm 0.114 & 0.007 \pm 0.112 & 0.011 \pm 0.140 \\ 0.006 \pm 0.110 & -0.352 \pm 0.112 & -0.004 \pm 0.103 \\ 0.015 \pm 0.124 & 0.006 \pm 0.120 & 0.215 \pm 0.124 \end{pmatrix}$
E	-1.280 ± 0.274	-0.837 ± 0.201
$R_{\rm CHSH}$	1.035 ± 0.161	0.717 ± 0.127

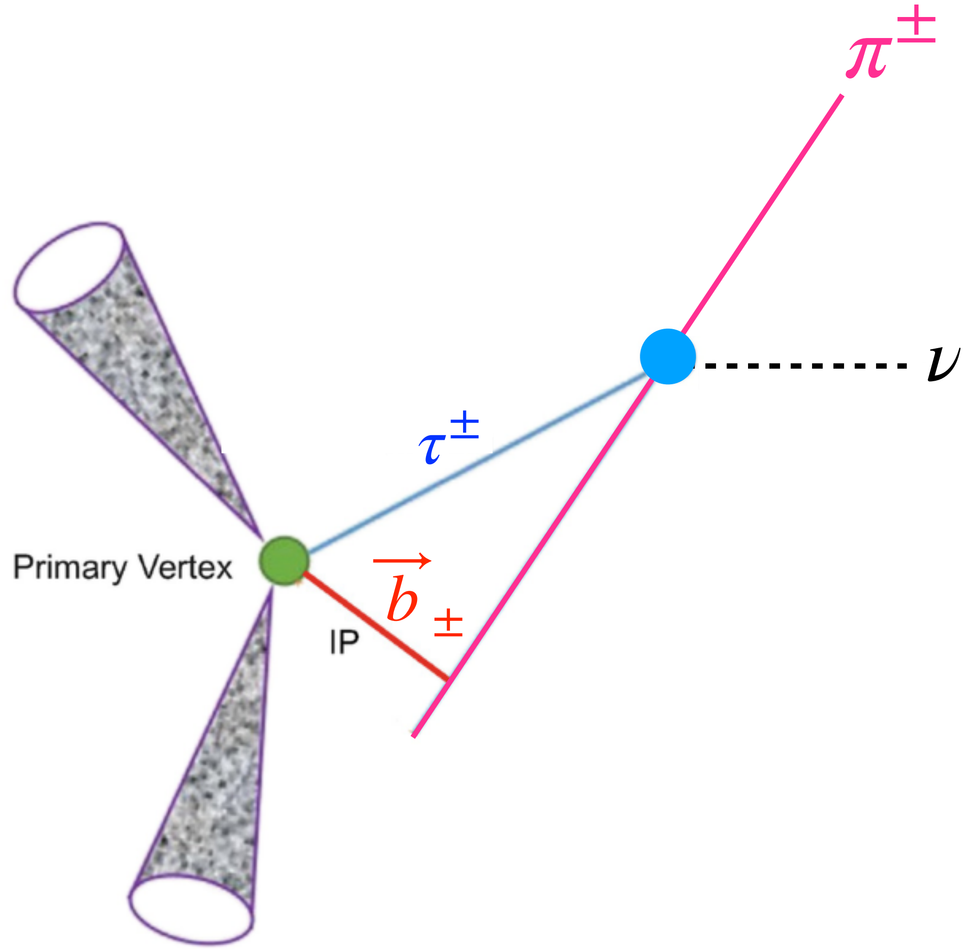
- The result is catastrophic. It may be blamed to the detector effect, since the reconstruction of tau-rest frames is very sensitive to the energy resolution.

SM values:

$$C_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$E = 3 \qquad \textbf{Entanglement} \implies E > 1$$

$$R_{\rm CHSH} = \sqrt{2} \simeq 1.414 \qquad \textbf{Bell-nonlocal} \implies R_{\rm CHSH} > 1$$



Use impact parameter information

- We use the information of impact parameter $\vec{b}_\pm$ measurement of $\pi^\pm$ to "correct" the observed energies of $\tau^\pm$ and Z decay products
- We check whether the reconstructed τ momenta are consistent with the measured impact parameters.
- We construct the likelihood function and search for the most likely τ momenta.

$$\boxed{E_\alpha(\delta_\alpha) = (1 + \sigma_\alpha^E \cdot \delta_\alpha) \cdot E_\alpha^{\text{obs}}}$$

$$\vec{b}_+ = |\vec{b}_+| \left(\sin^{-1}\Theta_+ \cdot \vec{e}_{\tau^+} - \tan^{-1}\Theta_+ \cdot \vec{e}_{\pi^+}\right)$$

$$\vec{\Delta}^i_{b_+}(\{\delta\}) \equiv \vec{b}_+ - |\vec{b}_+| \left(\sin^{-1}\Theta^i_+(\{\delta\}) \cdot \vec{e}^{\,i}_{\tau^+}(\{\delta\}) - \tan^{-1}\Theta^i_+(\{\delta\}) \cdot \vec{e}_{\pi^+}\right)$$

$$L^i_\pm(\{\delta\}) = \frac{[\Delta^i_{b_\pm}(\{\delta\})]^2_x + [\Delta^i_{b_\pm}(\{\delta\})]^2_y}{\sigma^2_{b_T}} + \frac{[\Delta^i_{b_\pm}(\{\delta\})]^2_z}{\sigma^2_{b_z}}$$

$$\boxed{L^i(\{\delta\}) = L^i_+(\{\delta\}) + L^i_-(\{\delta\})}$$

Results

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} 0.7803 \pm 0.195 & 0.019 \pm 0.162 & 0.046 \pm 0.180 \\ -0.001 \pm 0.171 & 0.858 \pm 0.165 & 0.000 \pm 0.178 \\ -0.024 \pm 0.188 & -0.010 \pm 0.162 & -0.678 \pm 0.184 \end{pmatrix}$	$\begin{pmatrix} 0.925 \pm 0.131 & -0.001 \pm 0.122 & 0.023 \pm 0.109 \\ 0.014 \pm 0.128 & 0.968 \pm 0.128 & -0.018 \pm 0.121 \\ -0.009 \pm 0.131 & -0.009 \pm 0.131 & -0.928 \pm 0.126 \end{pmatrix}$
E	2.182 ± 0.309	2.797 ± 0.191
$\mathcal{S}[\rho]$	1.626 ± 0.187	1.922 ± 0.155
R_{CHSH}	0.821 ± 0.167	1.273 ± 0.093

SM values: $C_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$E = 3$ **Entanglement** $\Longrightarrow E > 1$

$\mathcal{S}[\rho] = 2$ **Steerablity** $\Longrightarrow \mathcal{S}[\rho] > 1$

$R_{\text{CHSH}} = \sqrt{2} \simeq 1.414$ **Bell-nonlocal** $\Longrightarrow R_{\text{CHSH}} > 1$

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E	2.182 ± 0.309	$\sim 4\sigma$	2.797 ± 0.191	$\gg 5\sigma$
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Superiority of FCC-ee over ILC is due to a better beam resolution

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$

CP measurement

- Under CP, the spin correlation matrix transforms: $C \xrightarrow{CP} C^T$
- This can be used for a *model-independent* test of CP violation. We define:

$$A \equiv (C_{rn} - C_{nr})^2 + (C_{nk} - C_{kn})^2 + (C_{kr} - C_{rk})^2 \;\; \geq 0$$

- Observation of $A \neq 0$ immediately confirms CP violation.

- From our simulation, we observe

$$A = \begin{cases} 0.204 \pm 0.173 & \text{(ILC)} \\ 0.112 \pm 0.085 & \text{(FCC-ee)} \end{cases} \longleftarrow$$

consistent with absence of CPV

- This model independent bounds can be translated to the constraint on the CP-phase δ

$$\mathscr{L}_{\rm int} \propto H \bar{\psi}_\tau (\cos\delta + i\gamma_5 \sin\delta)\, \psi_\tau \;\Rightarrow\; C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \;\Rightarrow\; A(\delta) = 4\sin^2 2\delta$$

CP measurement

- Focusing on the region near $|\delta| = 0$, we find the 1-σ bounds:

$$|\delta| < \begin{cases} 8.9^o & \text{(ILC)} \\ 6.4^o & \text{(FCC-ee)} \end{cases}$$

- Other studies:

$$\Delta\delta \sim 11.5^o \quad \text{(HL-LHC)}$$ [Hagiwara, Ma, Mori 2016]

$$\Delta\delta \sim 4.3^o \quad \text{(ILC)}$$ [Jeans and G. W. Wilson 2018]

Summary

- High energy tests of entanglement and Bell inequality has recently attracted an attention.
- $\tau^+\tau^-$ pairs from $H \to \tau^+\tau^-$ form the EPR triplet state $|\Psi^{(1,0)}\rangle = \frac{|+,-\rangle + |-,+\rangle}{\sqrt{2}}$, and maximally entangled.
- We investigated feasibility of quantum property tests @ ILC and FCC-ee.
- Quantum test requires to a precise reconstruction of the tau rest frames and IP information is crucial to achieve this.
- Spin correlation is sensitive to CP-phase and we can measure the CP-phase as a byproduct of the quantum property measurement.

	Entanglement	Steering	Bell-inquality	CP-phase
ILC	$\sim 4\sigma$	$\sim 3\sigma$		8.9^o
FCC-ee	$\gg 5\sigma$	$\sim 5\sigma$	$\sim 3\sigma$	6.4^o

The research leading to the results presented in this talk has received funding from the Norwegian Financial Mechanism for years 2014-2021, grant nr 2019/34/H/ST2/00707

Understanding the Early Universe:
interplay of theory and collider experiments

Joint research project between the University of Warsaw & University of Bergen

$$
\begin{aligned}
\sigma(e^+e^- \to HZ)\big|_{\sqrt{s}=240\mathrm{GeV}} &= 240.3\,\mathrm{fb} \\
BR(H \to \tau^+\tau^-) &= 0.0632 \\
BR(\tau^- \to \pi^-\nu_\tau) &= 0.109 \\
BR(Z \to jj, \mu\mu, ee) &= 0.766 \\
\sigma(e^+e^- \to HZ)^{\mathrm{unpol}}_{240} \cdot BR_{H\to\tau\tau} \cdot [BR_{\tau\to\pi\nu}]^2 \cdot BR_{Z\to jj,\mu\mu,ee} &= 0.1382\,\mathrm{fb}
\end{aligned}
$$

Bell inequality

$$\left\langle s_a^\alpha \cdot s_b^\beta \right\rangle = \hat{a}_i \hat{b}_j \cdot \left\langle s_i^\alpha \cdot s_j^\beta \right\rangle = \hat{a}_i C_{ij} \hat{b}_i$$

unit vectors: $\hat{a}, \hat{a}', \hat{b}, \hat{b}'$

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2} \left| \left\langle s_a^\alpha \cdot s_b^\beta \right\rangle - \left\langle s_a^\alpha \cdot s_{b'}^\beta \right\rangle + \left\langle s_{a'}^\alpha \cdot s_b^\beta \right\rangle + \left\langle s_{a'}^\alpha \cdot s_{b'}^\beta \right\rangle \right|$$

$$= \frac{1}{2} \left| \hat{a}_i C_{ij} (\hat{b} - \hat{b}')_j + \hat{a}'_i C_{ij} (\hat{b} + \hat{b}')_j \right|$$

$$\max_{\hat{a}, \hat{a}', \hat{b}, \hat{b}'} \left[R_{\mathrm{CHSH}} \right] = \sqrt{\lambda_1 + \lambda_2}$$

($\lambda_1 \geq \lambda_2 \geq \lambda_3$ are 3 eigenvalues of $C^T C$)

Violation of Bell inequality implies

$$\boxed{\sqrt{\lambda_1 + \lambda_2} > 1}$$

M. Fabbrichesi, R. Floreanini, G. Panizzo (2021)