

Critical look at the time–energy uncertainty relations

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Introduction

The famous Heisenberg uncertainty relations [1] play an important and significant role in the understanding of the quantum world and in explanations of its properties. There is a mathematically rigorous derivation of the position–momentum uncertainty relation but this same can not be said about time–energy uncertainty relation. Nonetheless the time–energy uncertainty relation is considered by many authors as having the same status as the position–momentum uncertainty relation and it is often used as the basis for drawing far–reaching conclusions regarding the prediction of the behavior of some physical systems in certain situations in various areas of physics and astrophysics and from time to time such conclusions were considered as the crucial. So, the time–uncertainty relation still requires its analysis and checking whether it is correct and well motivated by postulates of quantum mechanics. We present here an analysis of the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations and show that the time–energy uncertainty relation can not be considered as universally valid.

Preliminaries: Uncertainty principle

The uncertainty principle belongs to the one of characteristic and the most important consequences of the quantum mechanics. The most known form of this principle is the Heisenberg uncertainty principle [1] for the position and momentum, which can be written as follows [2],

$$\Delta_{\phi}x \cdot \Delta_{\phi}p_x \geq \frac{\hbar}{2}, \quad (1)$$

where $\Delta_{\phi}x$ and $\Delta_{\phi}p_x$ are the standard (root-mean-square) deviations: In the general case for an observable F the standard deviation is defined as follows $\Delta_{\phi}F = \|(F - \langle F \rangle_{\phi} \mathbb{I})|\phi\rangle\|$, (where $\langle F \rangle_{\phi} \stackrel{\text{def}}{=} \langle \phi | F | \phi \rangle$ is the average (or expected) value of an observable F in a system whose state is represented by the normalized vector $|\phi\rangle \in \mathcal{H}$), provided that

$|\langle \phi | F | \phi \rangle| < \infty$. Equivalently: $\Delta_{\phi}F \equiv \sqrt{\langle F^2 \rangle_{\phi} - \langle F \rangle_{\phi}^2}$. (In Eq. (1) F stands for position and momentum operators x and p_x as well as for their squares). The observable F is represented by hermitian operator F acting in a Hilbert space \mathcal{H} of states $|\phi\rangle$. In general, the relation (1) results from basic assumptions of the quantum theory and from the geometry of Hilbert space [3].

Analogous relations hold for any two observables, say A and B , represented by non-commuting hermitian operators A and B acting in the Hilbert space of states (see [4]), such that $[A, B]$ exists and $|\phi\rangle \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, ($\mathcal{D}(\mathcal{O})$ denotes the domain of an operator \mathcal{O} or of a product of operators):

$$\Delta_{\phi} A \cdot \Delta_{\phi} B \geq \frac{1}{2} |\langle [A, B] \rangle_{\phi}|. \quad (2)$$

The inequality (2) is rigorous and its derivation simple. Indeed, defining $\delta A = A - \langle A \rangle_{\phi} \mathbb{I}$ and $\delta B = B - \langle B \rangle_{\phi} \mathbb{I}$, one finds that

$$[A, B] \equiv [\delta A, \delta B]. \quad (3)$$

Hence for all $|\phi\rangle \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$,

$$\begin{aligned}
|\langle \phi | [A, B] | \phi \rangle|^2 &\equiv |\langle \phi | [\delta A, \delta B] | \phi \rangle|^2 \\
&\equiv |\langle \phi | \delta A \delta B | \phi \rangle - \langle \phi | \delta B \delta A | \phi \rangle|^2 \\
&= |\langle \phi | \delta A \delta B | \phi \rangle - (\langle \phi | \delta A \delta B | \phi \rangle)^*|^2 \\
&= 4 |\operatorname{Im}. [\langle \phi | \delta A \delta B | \phi \rangle]|^2 \\
&\leq 4 |\langle \phi | \delta A \delta B | \phi \rangle|^2 \\
&\leq 4 \|\delta A | \phi \rangle\|^2 \cdot \|\delta B | \phi \rangle\|^2 \\
&\equiv 4 (\Delta_\phi A)^2 \cdot (\Delta_\phi B)^2,
\end{aligned} \tag{4}$$

which reproduces inequality (2). The above derivation seems to be the simplest one. (There is $(\Delta_\phi A)^2 \equiv \|\delta A | \phi \rangle\|^2$ and $(\Delta_\phi B)^2 \equiv \|\delta B | \phi \rangle\|^2$). One can find in the literature and in many textbooks the other derivations of the Heisenberg inequality. For completeness they are presented below in short.

In general one can meet a few methods of deriving the uncertainty relation. The most commonly used methods, which can be found in the literature, are the following: The first one follows the Robertson [4] and Schrödinger [5]. This method uses the obvious relation resulting from the Schwartz inequality,

$$\|\delta A|\phi\rangle\|^2 \|\delta B|\phi\rangle\|^2 \geq |\langle\phi|\delta A \delta B|\phi\rangle|^2. \quad (5)$$

The next step within this method is (see, e.g., [2, 3, 6, 7] and so on) to write the product $\delta A \delta B$ as a combination of the hermitian and anti-hermitian parts,

$$\delta A \delta B = \frac{\delta A \delta B + \delta B \delta A}{2} + i \frac{(-i)(\delta A \delta B - \delta B \delta A)}{2}. \quad (6)$$

Here

$$\{(-i)(\delta A \delta B - \delta B \delta A)\}^+ = [(-i)(\delta A \delta B - \delta B \delta A)] \equiv (-i)[A, B] \quad (7)$$

is the hermitian operator. Hence

$$|\langle\phi|\delta A \delta B|\phi\rangle|^2 = \frac{1}{4} |\langle\phi|(\delta A \delta B + \delta B \delta A)|\phi\rangle|^2 + \frac{1}{4} |\langle\phi|[A, B]|\phi\rangle|^2. \quad (8)$$

Now in order to obtain the desired result one should ignore the first component of the right hand side of the above equation which leads to the following inequality

$$|\langle \phi | \delta A \delta B | \phi \rangle|^2 \geq \frac{1}{4} |\langle \phi | [A, B] | \phi \rangle|^2 \quad (9)$$

and finally

$$\|\delta A|\phi\rangle\|^2 \|\delta B|\phi\rangle\|^2 \geq |\langle \phi | \delta A \delta B | \phi \rangle|^2 \geq \frac{1}{4} |\langle \phi | [A, B] | \phi \rangle|^2, \quad (10)$$

which is the proof of the inequality (2).

The similar simpler version of the above proof can be found eg. in [8, 9]. This proof also makes use of the inequality (5) but the estimation of the right hand side of this inequality is simpler. Namely within this proof expression $|\langle \phi | \delta A \delta B | \phi \rangle|$ is written as follows

$$|\langle \phi | \delta A \delta B | \phi \rangle|^2 = [\text{Re}.(\langle \phi | \delta A \delta B | \phi \rangle)]^2 + [\text{Im}.(\langle \phi | \delta A \delta B | \phi \rangle)]^2. \quad (11)$$

Next ignoring the the contribution coming from the real part of the above expression one obtains

$$|\langle \phi | \delta A \delta B | \phi \rangle|^2 \geq [\text{Im}(\langle \phi | \delta A \delta B | \phi \rangle)]^2 \quad (12)$$

but

$$\begin{aligned} \text{Im}(\langle \phi | \delta A \delta B | \phi \rangle) &= \frac{1}{2i} (\langle \phi | \delta A \delta B | \phi \rangle - \langle \phi | \delta A \delta B | \phi \rangle^*) \\ &= \frac{1}{2i} (\langle \phi | \delta A \delta B | \phi \rangle - \langle \phi | \delta B \delta A | \phi \rangle) \\ &\equiv \frac{1}{2i} \langle \phi | [A, B] | \phi \rangle. \end{aligned} \quad (13)$$

Thus

$$|\langle \phi | \delta A \delta B | \phi \rangle|^2 \geq \frac{1}{4} |\langle \phi | [A, B] | \phi \rangle|^2. \quad (14)$$

This result together with (5) means that

$$\|\delta A | \phi \rangle\|^2 \|\delta B | \phi \rangle\|^2 \geq \frac{1}{4} |\langle \phi | [A, B] | \phi \rangle|^2, \quad (15)$$

which is equivalent to the inequality (2).

The another method to proof the Heisenberg inequality one can find eg. in [10, 11]. Within this method using selfadjoint operators δA and δB one builds a new non-selfadjoint operator

$$L_\lambda = \delta A + i\lambda \delta B \neq L_\lambda^\dagger \quad (16)$$

where $\lambda = \lambda^* \geq 0$. Then

$$\begin{aligned} \|L_\lambda|\phi\rangle\|^2 &= \|\delta A|\phi\rangle\|^2 + i\lambda (\langle\phi|(\delta A\delta B - \delta B\delta A)|\phi\rangle) + \lambda^2 \|\delta B|\phi\rangle\|^2 \\ &\equiv \lambda^2 \|\delta B|\phi\rangle\|^2 + \lambda \langle\phi|(+i[A, B])|\phi\rangle + \|\delta A|\phi\rangle\|^2 \geq 0. \end{aligned} \quad (17)$$

Note that $(+i[A, B]) = (+i[A, B])^\dagger$ therefore the average value of $(+i[A, B])$ is a real number: $\langle\phi|(+i[A, B])|\phi\rangle = (\langle\phi|(+i[A, B])|\phi\rangle)^*$. So we have the second-degree polynomial in λ , which is positive for any λ . This implies that discriminant of this equation can not be a positive

$$(\langle\phi|(+i[A, B])|\phi\rangle)^2 - 4 \|\delta A|\phi\rangle\|^2 \|\delta B|\phi\rangle\|^2 \leq 0, \quad (18)$$

which again reproduces the inequality (2).

Summarizing the above part of the analysis of the Heisenberg uncertainty relations, let us note that from (5) it follows that the equality in the uncertainty relation (2) takes place when $|\phi\rangle$ is an eigenvector of A or B or when vectors $\delta A|\phi\rangle$ and $\delta B|\phi\rangle$ are parallel: $\delta A|\phi\rangle \parallel \delta B|\phi\rangle$ but from (11) it follows that this equality is possible only if additionally $\text{Re}(\langle\phi|\delta A \delta B|\phi\rangle) = 0$. From these conditions the following conclusion results: The necessary and sufficient condition for the minimum uncertainty, that is for the equality in the uncertainty relation (2) is

$$\delta B|\phi\rangle = i\kappa \delta A|\phi\rangle, \quad (19)$$

where $\kappa \in \mathbb{R}$. In particular for $A = x$ and $B = p$, that is for the position–momentum uncertainty relation, solutions of the criterion (19) are the so-called *coherent states*, which have the Gaussian form.

The defect, or perhaps the weakness, of methods (5) — (18) of deriving of the uncertainty relation (2) is that using them one should know in advance what a result should be obtained and which components appearing in the intermediate equations during the derivation process should be ignored. For example following the method (5) — (10) in order to obtain the desired result one should know earlier that the hermitian part in (8) should be ignored. Using the approach described by relations (11) — (15) one should know in advance that the result (2) can be obtained by ignoring in (11) the contribution coming from the real part of scalar product $\langle \phi | \delta A \delta B | \phi \rangle$. Finally using the method analyzed in (16) — (18) one should guess the form of such an auxiliary operator L_λ (see (16)), and then one should know in advance that in order to proof the relation (2) the $\|L_\lambda|\phi\rangle\|^2$ should be considered. Comparing methods described in relations (5) — (18) with the method (4) one can see that the derivation (4) of the relation (2) is much shorter and simpler than the others and it does not need any hidden assumption making in advance to find a desired result.

Analysis of the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations

Heisenberg in [1] postulated also the validity of the analogous relation to (1) for the time and energy (see also [12]). This relation was a result of his heuristic considerations and it is usually written as follows

$$\Delta_{\phi} t \cdot \Delta_{\phi} E \geq \frac{\hbar}{2}. \quad (20)$$

The more rigorous derivation of this relation was given by Mandelstam and Tamm [13] and now it is known as the Mandelstam–Tamm time–energy uncertainty relation. Their derivation is reproduced in [2] and goes as follows: In the general relation (2) the operator B is replaced by the selfadjoint non–depending on time Hamiltonian H of the system considered and $\Delta_{\phi} B$ is replaced by $\Delta_{\phi} H$ and then identifying the standard deviation $\Delta_{\phi} H$ with $\Delta_{\phi} E$ one finds that

$$\Delta_{\phi} A \cdot \Delta_{\phi} E \geq \frac{1}{2} |\langle [A, H] \rangle_{\phi}|, \quad (21)$$

where it is assumed that A does not depend upon the time t explicitly, $|\phi\rangle \in \mathcal{D}(HA) \cap \mathcal{D}(AH)$, and $[A, H]$ exists.

The next step is to use the Heisenberg representation and corresponding equation of motion which allows to replace the average value of the commutator standing in the right-hand side of the inequality (21) by the derivative with respect to time t of the expected value of A ,

$$\langle [A, H] \rangle_{\phi} \equiv i\hbar \frac{d}{dt} \langle A \rangle_{\phi}. \quad (22)$$

Using this relation one can replace the inequality (21) by the following one,

$$\Delta_{\phi} A \cdot \Delta_{\phi} E \geq \frac{\hbar}{2} \left| \frac{d}{dt} \langle A \rangle_{\phi} \right|. \quad (23)$$

(Relations (21) — (23) are rigorous). Next authors of [2, 13] and many others divide both sides of the inequality (23) by the term $\left| \frac{d}{dt} \langle A \rangle_{\phi} \right|$, which leads to the following relation

$$\frac{\Delta_{\phi} A}{\left| \frac{d}{dt} \langle A \rangle_{\phi} \right|} \cdot \Delta_{\phi} E \geq \frac{\hbar}{2}, \quad (24)$$

or, using

$$\tau_A \stackrel{\text{def}}{=} \frac{\Delta_\phi A}{\left| \frac{d}{dt} \langle A \rangle_\phi \right|}, \quad (25)$$

they come to the final result known as the Mandelstam–Tamm time–energy uncertainty relation,

$$\tau_A \cdot \Delta_\phi E \geq \frac{\hbar}{2}, \quad (26)$$

where τ_A is usually considered as a time characteristic of the evolution of the statistic distribution of A [2]. The time–energy uncertainty relation (26) and the above described derivation of this relation is accepted by many authors analyzing this problem or applying this relation (see, e.g. [14, 15, 16, 17] and many other papers). On the other hand there are some formal controversies regarding the role and importance of the parameter τ_A in (26) or Δt in (20).

These controversies are caused by the fact that in the quantum mechanics the time t is a parameter. Simply it can not be described by the hermitian operator, say T , acting in the Hilbert space of states (that is time can not be an observable) such that $[H, T] = i\hbar\mathbb{I}$ if the Hamiltonian H is bounded from below. This observation was formulated by Pauli [18] and it is known as "Pauli's Theorem" (see, eg. [14, 19]). Therefore the status of the relations (20) and relations (1), (2) is not the same regarding the basic principles of the quantum theory (see also discussion, e.g., in [20, 21, 22, 23]).

The Pauli's conclusion follows from the following analysis: If $T = T^+$ and $[H, T] = i\hbar\mathbb{I}$ then

$$[H, T^n] = i\hbar n T^{n-1} \equiv i\hbar \frac{\partial}{\partial T} T^n. \quad (27)$$

From this last relation it follows that

$$\left[H, e^{-i\frac{\lambda}{\hbar} T} \right] = \lambda e^{-i\frac{\lambda}{\hbar} T}. \quad (28)$$

The consequence of the above commutation relations is that for all $\lambda \in \mathbb{R}$

$$e^{+i\frac{\lambda}{\hbar}T} H e^{-i\frac{\lambda}{\hbar}T} = H + \lambda \mathbb{I} \stackrel{\text{def}}{=} H_\lambda. \quad (29)$$

From the Stone theorem we know that if T is a selfadjoint operator, $T = T^+$, then the operator

$$U_\lambda \stackrel{\text{def}}{=} e^{+i\frac{\lambda}{\hbar}T}, \quad (30)$$

is the unitary operator: $U_\lambda U_\lambda^+ = U_\lambda^+ U_\lambda = \mathbb{I}$. So, operators H and H_λ are unitarily equivalent and hence they must have the same spectrum. From (29) it follows that they commute: $[H, H_\lambda] = 0$ and thus they have common eigenfunctions. The spectrum of H_λ ranges over the whole real line \mathbb{R} (see, e.g. [19]) but, by assumption, the spectrum of H is bounded from below. Therefore H_λ and H can not be unitarily equivalent. In other words, the operator U_λ can not be the unitary operator. Hence the conclusion that the operator T defining the operator U_λ can not be a selfadjoint operator.

The Mandelstam–Tamm uncertainty relation (26) is also not free of controversies. People applying and using the above described derivation of (26) in their discussions of the time–energy uncertainty relation made use (consciously or not) of a hidden assumption that right hand sides of Eqs. (21), (23) are non-zero, that is that there does not exist any vector $|\phi_\beta\rangle \in \mathcal{H}$ such that $\langle [A, H] \rangle_{\phi_\beta} = 0$, or $\frac{d}{dt} \langle A \rangle_{\phi_\beta} = 0$. Although in the original paper of Mandelstam and Tamm [13] there is a reservation that for the validity of the formula of the type (26) it is necessary that $\Delta_\phi H \neq 0$ (see also, e.g. [24, 25]), there are not an analogous reservations in [2] and in many other papers.

Basic principles of mathematics require that before the dividing the both sides of Eq. (23) by $|\frac{d}{dt} \langle A \rangle_\phi|$, one should check whether $\frac{d}{dt} \langle A \rangle_\phi$ is different from zero or not. Let us do this now: Let $\Sigma_H \subset \mathcal{H}$ be a set of eigenvectors $|\phi_\beta\rangle$ of H for the eigenvalues E_β . We have $H|\phi_\beta\rangle = E_\beta|\phi_\beta\rangle$ for all $|\phi_\beta\rangle \in \Sigma_H$ and therefore for all $|\phi_\beta\rangle \in \Sigma_H \cap \mathcal{D}(A)$ (see (22)),

$$\langle [A, H] \rangle_{\phi_\beta} = i\hbar \frac{d}{dt} \langle A \rangle_{\phi_\beta} \equiv 0. \quad (31)$$

Similarly,

$$\Delta_{\phi_\beta} H = \sqrt{\langle |H^2| \rangle_{\phi_\beta} - (\langle |H| \rangle_{\phi_\beta})^2} \stackrel{\text{def}}{=} \Delta_{\phi_\beta} E \equiv 0, \quad (32)$$

for all $|\phi_\beta\rangle \in \Sigma_H$. This means that in all such cases the non-strict inequality (23) takes the form of the following equality

$$\Delta_\phi A \cdot 0 = \frac{\hbar}{2} \cdot 0. \quad (33)$$

In other words, one can not divide the both sides of the inequality (23) by $|\frac{d}{dt}\langle A \rangle_\phi| \equiv 0$ for all $|\phi_\beta\rangle \in \Sigma_H$, because in all such cases the result is an undefined number and such mathematical operations are unacceptable. It should be noted that although the authors of the publications [2, 24] and many others knew that the property (31) occurs for the vectors from the set Σ_H , it did not prevent them to divide both sides of the inequality (23) by $|\frac{d}{dt}\langle A \rangle_\phi|$, that is by $|\frac{d}{dt}\langle A \rangle_\phi| \equiv 0$, without taking into account (32) and without any explanations.

What is more, this shows that there is no reason to think of τ_A as infinity in this case as it was done, e.g., in [2, 24]. In general, **the problem is that usually the set Σ_H of the eigenvectors of the Hamiltonian H is a linearly dense (complete) set in the state space \mathcal{H} .** Hence the conclusion that such relations as (24) and then (26) can not be considered as correct and rigorous seems to be justified. Summing up, we have proved that contrary to the uncertainty relations (1) and (2), the relations of type (20) and (26) can not hold on linearly dense sets in the state space \mathcal{H} and therefore such relations can not be considered as the universally valid.

Discussion

The above formulated conclusion agrees with the intuitive understanding of stationary states. The stationary states are represented by eigenvectors of the Hamiltonian H of the system considered and if it is known that the system is in a stationary state represented, say, by the state vector $|\phi_\beta\rangle$ and then $\langle E \rangle_\phi \equiv \langle H \rangle_\phi = E_\beta$ and $\Delta_{\phi_\beta} E = \Delta_{\phi_\beta} H = 0$ and then one is sure that at any time t (and during any time interval $\Delta t = t_2 - t_1$, where $t_1 < t_2 < \infty$) the energy is equal E_β or that $\Delta E = E_\beta(t_2) - E_\beta(t_1) \equiv 0$. Similar picture one meets when $|\phi\rangle = |\phi_\alpha\rangle$ is an eigenvector for A . (This case was also noticed in [24]). Then also for any $|\phi_\alpha\rangle \in \Sigma_A \cap \mathcal{D}(H)$, (where by Σ_A we denote the set of eigenvectors $|\phi_\alpha\rangle$ for A), $|\frac{d}{dt}\langle A \rangle_\phi| \equiv 0$ and $\Delta_\phi A \equiv 0$. Thus, instead of (33) one once more has $0 \cdot \Delta_\phi H = \frac{\hbar}{2} \cdot 0$, and once again dividing both sides of this inequality by zero has no mathematical sense. Now note that the relations (1), (2) are always satisfied for all $|\phi\rangle \in \mathcal{H}$ fulfilling the conditions specified before Eq. (2), and in contrast to this property, we have proved that the Mandelstam–Tamm relation (24) can not be true not only on the set $\Sigma_H \subset \mathcal{H}$, whose span is usually dense in \mathcal{H} , but also on the set $\Sigma_A \subset \mathcal{H}$.

A detailed analysis of relations (20), (26) suggests that they may be in conflict with one of the basic postulates of Quantum Mechanics: Namely, with the projection (reduction) postulate. It is because the projection postulate leads to the Quantum Zeno Effect [26] (see also, e.g., [27, 28, 29, 30]), that is it makes possible to force the system to stay in a given state as a result of continuous or quasi–continuous observations verifying if the system is in this given state. It is possible if time interval separating the successive measurements (observations) are separated by suitable short time intervals Δt such that $\Delta t \rightarrow 0$ when the number of observations increases [27, 28, 29, 30]. In general the duration of each of these measurements must be shorter than the time interval separating them, and in turn, the uncertainty of the time t can not be larger than the duration of these measurements. Therefore the conclusion that the relation (20) should make impossible to observe the Quantum Zeno Effect seems to be legitimate. Contrary to such a conclusion there are experimental tests verifying and confirming this effect [31].

The state of the system is characterized by a set of quantum numbers and one of these numbers is the energy of the system in the state considered. Therefore if the quantum system is forced to stay in the given state by continuously or quasi-continuously checking it if it is in this state, then quantum numbers characterizing this state (including the energy) also remain unchanged. This means that there is $\Delta E = 0$ and $\Delta t \rightarrow 0$ in such a case and thus there is a conflict with relations (1), (26).

As it was mentioned earlier there is a reservation in [13] that derivation of (26) does not go for eigenvectors of H (Then $\Delta H = 0$). In fact it can be only applied for eigenvectors corresponding to the continuous part of the spectrum of H . As an example of possible applications of the relation (26) unstable states modeled by wave-packets of such eigenvectors of H are considered in [13], where using (26) the relation connecting half-time $\tau_{1/2}$ of the unstable state, say $|\varphi\rangle$, with the uncertainty $\Delta_\varphi H$ was found: $\tau_{1/2} \cdot \Delta_\varphi H \geq \frac{\pi}{4} h$. In general, when one considers unstable states such a relation and the similar one appear naturally [32, 33, 34] but this is quite another situation than that described by the relations (1), (2).

The other example is a relation between a life–time τ_φ of the system in the unstable state, $|\varphi\rangle$, and the decay width Γ_φ : In such cases we have $\tau_\varphi \cdot \Gamma_\varphi = \hbar$ but there are not any uncertainties of the type ΔE and Δt in this relation (see, e.g., [32]). Note that in all such cases the vector $|\varphi\rangle$ representing the unstable state can not be the eigenvector of the Hamiltonian H . It should be noted here that even in the case of unstable states one should be very careful using the relation (26): For example in the case of unstable states $|\phi\rangle$ modeled by the Breit–Wigner energy density distribution $\omega_{BW}(E) = \frac{N}{2\pi} \Theta(E - E_{min}) \frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}$, where $\Theta(E)$ is the unit step function and N is the normalization constant, the average values $\langle H \rangle_\phi = \int_{E_{min}}^{\infty} E \omega_{BW}(E) dE$ and $\langle H^2 \rangle_\phi = \int_{E_{min}}^{\infty} E^2 \omega_{BW}(E) dE$ have not definite values and hence $\Delta_\phi H$ is undefined which means that the relation (26) does not work in this case.

In addition to the doubts discussed above and relating to validity of the time–energy uncertainty relations a thorough analysis of the relation (20) suggests one more interpretative ambiguity. Namely let us consider the minimal uncertainty version of (20):

$$\Delta_{\phi} t \cdot \Delta_{\phi} E = \frac{\hbar}{2}. \quad (34)$$

Then, let us invoke a much older relation, namely the Planck–Einstein relation:

$$E_{\phi} = h\nu_{\phi}, \quad (35)$$

(where h is the Planck's constant and ν_{ϕ} is the frequency), which constituted one of the foundations enabling the emergence of quantum mechanics. This relation plays still a fundamental role in Quantum Theory and it was verified many times using direct and indirect methods.

The frequency ν_{ϕ} is connected with the period \mathcal{T}_{ϕ} by the relation

$$\nu_{\phi} = \frac{1}{\mathcal{T}_{\phi}}. \quad (36)$$

Using the last relation (36) one can rewrite the Planck–Einstein relation (35) as follows:

$$\mathcal{T}_\phi E_\phi = h, \quad (37)$$

which means that there is,

$$\mathcal{T}_\phi E_\phi > \frac{\hbar}{2}. \quad (38)$$

From the mathematical point of view equations (34) and (37) are identical. (To be more precise: the equation (37) is a re-scaled version of the equation (34) and scaling factor equals $\frac{1}{4\pi} \simeq 0.08$). On the other hand the inequality (38) is the strong case of the Heisenberg inequality (20). The problem is that relations (37) (and (35)) combine exact values of time $t = \mathcal{T}_\phi$ and energy E_ϕ (or E_ϕ and ν_ϕ) with each other while the equation (34) combines uncertainties of time t and energy E . In the light of this analysis the standard interpretation of the Heisenberg relation (34) (and (20)) may not be obvious and correct.

Equation (37) says that if one finds that the exact value of the energy of the photon in the state $|\phi\rangle$ is E_ϕ , then one is sure that the period is \mathcal{T}_ϕ (or that the frequency is $\nu_\phi = 1/\mathcal{T}_\phi$) and, of course because the value of E is exact then in this case there must be $\Delta_\phi E = 0$. At the same time, equation (34) and inequality (20) state that if the value of E is exact and thus $\Delta_\phi E = 0$ then simultaneously there must be $\Delta_\phi t = \infty$, which means that it should be impossible to determine the exact value of the period \mathcal{T}_ϕ or frequency ν_ϕ .






Conclusions

- The discussion of relations (20) and (26) presented in previous Sections and the detailed analysis of the derivation of the relation (26) suggests that these time–energy uncertainty relations are not well founded and can not be considered as universally valid.
- When using these relations as the basis for predictions of the properties and of a behavior of some systems in physics or astrophysics (including cosmology — see, e.g., [17, 35]) one should be very careful interpreting and applying results obtained.
- In some problems the use of the relation (26) may be reasonable (see, e.g. the case of unstable states) but then it should not be interpreted analogously to the relations (1), (2).

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Thank you for your attention

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